Two rotating black holes in the stationary axially symmetric case.

by

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Abstract

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The topic of this thesis is a two body problem in General Relativity in which space time is stationary and axially symmetric. Two rotating black holes are located on the axis of the symmetry. In the stationary axially symmetric case, the Einstein's vacuum equations are equivalent to the Ernst equation. A class of solutions for the Ernst equation is constructed using Lax pairs and the Riemann-Hilbert approach. A solution for the configuration in question is extracted from the general formula. The space time is reconstructed and the force acting between the bodies is computed as a flux of the spatial part of the stress-energy tensor through some two dimensional surface surrounding one of the bodies.

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Introduction.

The problem of two black holes in the stationary, axially symmetric case was first discussed in [BachWeyl]. They established that the configuration cannot be stable. Space time has a conical singularity on the axis between the bodies. The force acting between the bodies, defined as a flux of the stress energy tensor, comes from the flux through a surface transversal to the conical singularity. Physically one may think of the conical singularity as providing a "strut "which keeps two body system from collapsing. The work in [BachWeyl] is limited to the case with no rotation.

In this thesis we will investigate two rotating black holes (i.e. double Kerr solution). The goal is to construct space time, prove that it is physical, identify parameters, and compute the correction to the force due to rotation. It was conjectured early on that, for appropriate angular momenta, the spin-spin interaction of the bodies could balance their gravitational attraction, providing a stable configuration and eliminating the need for a "strut "(see also [Dietz]). Our formulae show that for small angular momenta this does not happen (again s.f. [Dietz]), and in fact the force **increases** with the spin (see formula (9.19)).

The existing literature [KraSte] offers a wide range of techniques to construct new solutions of the Einstein vaccuum equation (EVE) from ones already known. However, the physical meaning of the parameters and the physical realizability of the solution is not always evident. A succession of articles beginning with [Wein] contains a list of conditions that space time must satisfy in order to be physically realistic and uniquely specified. A nonconstructive prove of existence is also given [Wein].

The construction of space time involves solving a nonlinear PDE called the Ernst equation [Ernst]. There is a difference of opinion in the literature as to the choice of the unknown function (Ernst potential). In [Wein], the real part of the Ernst potential is the diagonal angular coefficient of the space time metric. For classical solutions, this function is nonsingular and nonvanishing except for the axis of symmetry. This is convenient for the application of PDE techniques [Wein]. However, the precise expressions arising from such a choice of the Ernst potential are too complex to analyze using explicit techniques as in [Korot],[Dietz],[BelZakh] and [BachWeyl]. The alternative choice of the Ernst potential is a diagonal time coefficient of the metric. Such a choice allows for rather easy identification of classical solutions among a general class of solutions. However, there are complications. For the Kerr solution, the time-time term of the metric vanishes at a surface, called the ergosphere, surrounding the body. The presence of an ergospere complicates the establishment of uniqueness, as well as a proof that any particular solution constructed on a base of such an Ernst potential is physical.

We will construct a solution to the two-rotating black hole problem by employing a slight modification of the technique presented in [Korot]. We compute the force acting between the bodies, outline a procedure for transforming between the two kinds of Ernst potentials and use the results of [Wein] to show that the constructed solution is a unique physical solution for the configuration in question.

The double Kerr solution is also considered in [Dietz] using a different approach and an alternative identification of the physical parameters. Our proof of uniqueness is, however, not completely rigorous and rests on some plausible assumptions about the nature of the solution.

CHAPTER 1

Axially symmetric stationary space time.

The purpose of this chapter is to explain the assumptions about the symmetries of the space time, to derive the most common form of space time metrics satisfying these symmetries and to specify an unknown function (Ernst potential) and the equation that it satisfies (Ernst equation). The results presented in this chapter are well known.

Suppose M is an n-dimensional differentiable manifold and $\{x^1, ..., x^n\}$ are local coordinates. We divide these coordinates into two sets, $\{x^1, ..., x^k\}$ and $\{x^{k+1}, ..., x^n\}$ and use capital Latin letters $A, B, ...$ to denote indices withint the range $1, ..., k$ and Greek letters $\alpha, \beta, ...$ to denote indices within the range $k+1, ..., n$. Lower case Latin letters $i, j, ...$ will refer to the range $1, ..., n$.

The following are two well known versions of the Frobenius Theorem [Fe-Cla].

Theorem 1.

Let $\{V_\alpha\}$ be vector fields independent at each point and satisfying

$$[V_\alpha, V_\beta] = C^\delta_{\alpha\beta} V_\delta$$

on M, for some functions $C^\delta_{\alpha\beta}$.

Then

1. In a neighborhood of any point, there exist coordinates $\{x^i\}$ such that

$$V_\alpha(x^A) = 0.$$

2. If in addition, the fields $\{V_\alpha\}$ commute, then $\{x^i\}$ may be chosen so that

$$V_\alpha(x^i) = \delta^i_\alpha.$$

Theorem 2.

Let $\overset{k+1}{\omega}, \ldots, \overset{n}{\omega}$ be 1-forms, independent at each point and satisfying

$$\overset{k+1}{\omega} \wedge \overset{k+2}{\omega} \wedge \ldots \wedge \overset{n}{\omega} \wedge d\overset{\alpha}{\omega} = 0.$$

Then there exists, near any point, a coordinate system $\{x^i\}$ such that

$$\overset{\alpha}{\omega}(\partial_A) = 0.$$

If a metric is given, then vector fields correspond to 1-forms according to the relation

$$\omega = g(V, .).$$

We are interested in the situation when we have a set of 1-forms $\{\overset{\alpha}{\omega}\}$ satisfying Theorem 2, as well as a set of vector fields $\{V_\alpha\}$ satisfying Theorem 1 and the vector fields correspond to the 1-forms. We denote by E_p a linear space spanned by $\{V_\alpha\}$, taken at point p. Let $\mathcal{L}_V$ denote the Lie derivative with respect to a vector field V.

Theorem 3.

Suppose a metric g is given and independent vector fields $\{V_\alpha\}$ are such that a restriction of g to E_p is non singular. Suppose further that $\{V_\alpha\}$ satisfy the *conditions of Theorem 1 and 1-forms corresponding to* $\{V_\alpha\}$ *satisfy the conditions of Theorem 2.*

Then there exists a coordinate system $\{x^i\}$ *such that*

1). $g_{\alpha B} = 0$, *for* $\alpha = k+1, ..., n$ *and* $B = 1, ..., k$.

2). $V^A_\alpha = 0$, *for* $A = 1, ..., k$ *and* $\alpha = k+1, ..., n$.

If, in addition, $\{V_\alpha\}$ *are Killing vector fields (which means that* $\mathcal{L}_{V_\alpha} g = 0$*), then*

$$\partial_\alpha g_{ij} = 0.$$

It is our assumption that the conditions of Theorem 3 are satisfied with $n = 4$, $k = 2$ and V_3 and V_4 are asymptotically space and time like Killing vector fields respectively. Thus, there exist coordinates $(t, \phi, x^A)_{A=1,2}$ such that

$$ds^2 = V dt^2 + 2W dt d\phi + X d\phi^2 + g_{AB} dx^A dx^B, \tag{1.1}$$

$A, B = 1, 2$, $V_3 = \partial_\phi$, $V_4 = \partial_t$, $\partial_\phi g_{ij} = 0$, $\partial_t g_{ij} = 0$, $V =< \partial_t, \partial_t >$, $W =< \partial_\phi, \partial_t >$ and $X =< \partial_\phi, \partial_\phi >$. At this stage, (1.1) is a local result. We make it global by assuming that M is equivalent to a principal bundle $M = R \otimes SO(2) \otimes N$, where N is a 2-dimensional differentiable manifold. The R refers to the integral lines of ∂_t and $SO(2)$ refers to the integral lines of ∂_ϕ. The set of space points which are not moved by the flow of ∂_ϕ is called an axis of ϕ symmetry. The described situation is called a stationary axially symmetric space time.

It can be shown [Wein] that a quantity $\rho > 0$, satisfying

$$\rho^2 = -VX + W^2 \neq 0, \tag{1.2}$$

is harmonic and therefore, has no saddle points in N. Consequently, it may be taken as a coordinate in N. The second coordinate z is taken orthogonal to ρ. Therefore, in coordinates (ρ, z), $g_{AB}dx^A dx^B = C(\rho, z)(d\rho^2 + dz^2)$ for some $C(\rho, z)$. This result is traditionally written in the form

$$ds^2 = e^{-2U}(e^{2K}(d\rho^2 + dz^2) + \rho^2 d\phi^2) - e^{2U}(dt + Ad\phi)^2$$

with functions $U = U(\rho, z)$, $K = K(\rho, z)$ and $A = A(\rho, z)$.

The computation of the Ricci tensor for a space-time given by the above expression is straightforward. The resulting set of vacuum equations is of the form $\{F_i(U, A, K) = 0\}_{i=1,...,5}$, where $F_i(...)$ involves partial derivatives of the arguments with respect to ρ and z, up to second order.

To present $\{F_i(U, A, K) = 0\}_{i=1,...,5}$ in a useful form, we introduce a function Φ according to the relations

$$\partial_\rho \Phi = -\frac{v^2}{\rho}\partial_z A,$$

$$\partial_z \Phi = \frac{v^2}{\rho}\partial_\rho A,$$

$$v \equiv e^{2U}.$$

$\{F_i(U, A, K) = 0\}_{i=1,...,5}$ contains a compatibility condition for these relations. One introduces the so called Ernst potential $\mathcal{E} = v + i\Phi$. After transformations, the equations $\{F_i(U, A, K) = 0\}$ are equivalent to the following:

$$(\mathcal{E} + \bar{\mathcal{E}})(\mathcal{E}_{zz} + \rho^{-1}\mathcal{E}_\rho + \mathcal{E}_{\rho\rho}) = 2(\mathcal{E}_\rho^2 + \mathcal{E}_z^2),$$

$$\partial_\rho K = \frac{\rho}{4v^2}(v_\rho^2 - v_z^2 + \Phi_\rho^2 - \Phi_z^2),$$

$$\partial_z K = \frac{\rho}{2v^2}(v_\rho v_z + \Phi_\rho \Phi_z),$$

and compatibility conditions for Φ.

Conversely, suppose that one has a solution $\mathcal{E}$ of the (Ernst) equation

$$(\mathcal{E} + \bar{\mathcal{E}})(\mathcal{E}_{zz} + \rho^{-1}\mathcal{E}_\rho + \mathcal{E}_{\rho\rho}) = 2(\mathcal{E}_\rho^2 + \mathcal{E}_z^2), \tag{1.3}$$

then one computes functions $U = U(\rho, z)$, $K = K(\rho, z)$ and $A = A(\rho, z)$ according to

$$e^{2U} \equiv v \equiv Re[\mathcal{E}], \tag{1.4}$$

$$\Phi \equiv Im[\mathcal{E}], \tag{1.5}$$

$$\partial_z A = -\frac{\rho}{v^2}\partial_\rho \Phi, \tag{1.6}$$

$$\partial_\rho A = \frac{\rho}{v^2}\partial_z \Phi, \tag{1.7}$$

$$\partial_\rho K = \frac{\rho}{4v^2}(v_\rho^2 - v_z^2 + \Phi_\rho^2 - \Phi_z^2), \tag{1.8}$$

$$\partial_z K = \frac{\rho}{2v^2}(v_\rho v_z + \Phi_\rho \Phi_z). \tag{1.9}$$

Boundary conditions at infinity

$$A(\rho, z) \to 0, \ \text{ as } \ (\rho, |\mathrm{z}|) \to \infty, \tag{1.10}$$

$$K(\rho, z) \to 0, \ \text{ as } \ (\rho, |\mathrm{z}|) \to \infty, \tag{1.11}$$

follow from the requirement of asymptotic flatness of space time. There are the boundary conditions at the axis of symmetry $\rho = 0$. These depend on the physical situation. We will concentrate on studying two rotating black holes. It will later be seen that in $\{\rho, z\}$ variables, the event horizon of each of the bodies collapses into a degenerated ellipse $\{\rho = 0, z \in [\lambda_i, \lambda_{i+1}]\}_{i=1,3}$, for some $\{\lambda_i\}$. As usual, the "event horizon "refers to a two dimensional closed surface that can be penetrated by light in only one direction. The event horizons split the z-axis into five parts. We require that A vanishes on those parts of the z-axis outside of the event horizons.

Such functions A, K and U give axially symmetric stationary space-time according to

$$ds^2 = e^{-2U}(e^{2K}(d\rho^2 + dz^2) + \rho^2 d\phi^2) - e^{2U}(dt + Ad\phi)^2. \tag{1.12}$$

The real and imaginary parts of the Ernst equation (1.3) are the compatibility conditions for (1.4)-(1.9).

Equations (1.3-1.12) appear in [KraSte].

CHAPTER 2

Lax pair for the Ernst equation.

The goal of this chapter is to show that as in [NeugCra], the system

$$\Psi_\xi = \left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + f \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \right] \Psi, \tag{2.1}$$

$$\Psi_{\overline{\xi}} = \left[\begin{pmatrix} \overline{B} & 0 \\ 0 & \overline{A} \end{pmatrix} + \frac{1}{f} \begin{pmatrix} 0 & \overline{B} \\ \overline{A} & 0 \end{pmatrix} \right] \Psi, \tag{2.2}$$

$$\Psi(\infty^+) = \begin{pmatrix} \overline{\mathcal{E}} & 1 \\ \mathcal{E} & -1 \end{pmatrix},$$

$$A = \frac{\overline{\mathcal{E}}_\xi}{\overline{\mathcal{E}} + \mathcal{E}}, \tag{2.3}$$

$$B = \frac{\mathcal{E}_\xi}{\overline{\mathcal{E}} + \mathcal{E}}, \tag{2.4}$$

$$f = f(\lambda, \xi, \overline{\xi}) = \sqrt{\frac{\lambda - i\overline{\xi}}{\lambda + i\xi}},$$

$$\xi = \rho + iz, \overline{\xi} = \rho - iz,$$

with spectral variable $\lambda \in C$, where $\Psi = \Psi(\lambda, \xi, \overline{\xi})$, $\mathcal{E} = \mathcal{E}(\xi, \overline{\xi})$, $A = A(\xi, \overline{\xi})$ and $B = B(\xi, \overline{\xi})$ is a Lax couple for the Ernst equation

$$(\mathcal{E} + \overline{\mathcal{E}})(\mathcal{E}_{zz} + \rho^{-1}\mathcal{E}_\rho + \mathcal{E}_{\rho\rho}) = 2(\mathcal{E}_\rho^2 + \mathcal{E}_z^2). \tag{2.5}$$

A Riemann surface of the function $f = f(\lambda)$ is regarded as two copies of the complex plane C glued in the standard way along the straight cut $[i\overline{\xi}, -i\xi]$. The notation ∞^+ refers to the infinity on the sheet where $f \to 1$ as $|\lambda| \to +\infty$.

Introducing

$$a = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.6}$$

we have that

$$\sigma a = \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}, a\sigma = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}, \sigma a \sigma = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix},$$

$$\Psi_\xi = (a + f a \sigma)\, \Psi,$$

$$\Psi_{\overline{\xi}} = \left(\sigma \overline{a} \sigma + \frac{1}{f} \sigma \overline{a}\right) \Psi,$$

or

$$\Psi_\xi = a\,(e + f\sigma)\, \Psi,$$

$$\Psi_{\overline{\xi}} = \sigma \overline{a} \left(\sigma + \frac{1}{f} e\right) \Psi,$$

where $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Now,

$$\Psi_{\xi\overline{\xi}} = a_{\overline{\xi}}\,(e + f\sigma)\, \Psi + a \left(f_{\overline{\xi}} \sigma\right) \Psi + a\,(e + f\sigma)\, \sigma \overline{a} \left(\sigma + \frac{1}{f} e\right) \Psi$$

and

$$\Psi_{\overline{\xi}\xi} = \sigma \overline{a}_\xi \left(\sigma + \frac{1}{f} e\right) \Psi + \sigma \overline{a} \left(\left(\frac{1}{f}\right)_\xi e\right) \Psi + \sigma \overline{a} \left(\sigma + \frac{1}{f} e\right) a\,(e + f\sigma)\, \Psi.$$

Thus, the compatibility condition is

$$\begin{aligned}
&a_{\overline{\xi}}\left(e+f\sigma\right)+f_{\overline{\xi}}a\sigma+a\left(e+f\sigma\right)\sigma\overline{a}\left(\sigma+\frac{1}{f}e\right)\\
=\;&\sigma\overline{a}_{\xi}\left(\sigma+\frac{1}{f}e\right)+\left(\frac{1}{f}\right)_{\xi}\sigma\overline{a}+\sigma\overline{a}\left(\sigma+\frac{1}{f}e\right)a\left(e+f\sigma\right),
\end{aligned}\tag{2.7}$$

$$f_{\overline{\xi}}=\left(\sqrt{\frac{\lambda-i\overline{\xi}}{\lambda+i\xi}}\right)_{\overline{\xi}}=\frac{1}{\sqrt{\lambda+i\xi}}\frac{1}{2\sqrt{\lambda-i\overline{\xi}}}(-i)$$

and

$$\left(\frac{1}{f}\right)_{\xi}=\left(\sqrt{\frac{\lambda+i\xi}{\lambda-i\overline{\xi}}}\right)_{\xi}=\frac{1}{\sqrt{\lambda-i\overline{\xi}}}\frac{1}{2\sqrt{\lambda+i\xi}}i.$$

If $\lambda\to\infty$, then (2.7) gives

$$\begin{aligned}
&a_{\overline{\xi}}\left(e+\sigma\right)+a\left(e+\sigma\right)\sigma\overline{a}\left(\sigma+e\right)\\
=\;&\sigma\overline{a}_{\xi}\left(\sigma+e\right)+\sigma\overline{a}\left(\sigma+e\right)a\left(e+\sigma\right).
\end{aligned}\tag{2.8}$$

If $\lambda\to -i\xi$, then

$$\begin{aligned}
&f_{\overline{\xi}}\sim-\frac{i}{2}\frac{1}{\sqrt{-i(\xi+\overline{\xi})}}\frac{1}{\sqrt{\lambda+i\xi}},\\
&f\sim\sqrt{-i(\xi+\overline{\xi})}\frac{1}{\sqrt{\lambda+i\xi}},\\
&\left(\frac{1}{f}\right)_{\xi}\sim\frac{i}{2}\frac{1}{\sqrt{-i(\xi+\overline{\xi})}}\frac{1}{\sqrt{\lambda+i\xi}}.
\end{aligned}$$

Evaluating the residues in (2.7), we obtain

$$a_{\overline{\xi}}\sqrt{-i(\xi+\overline{\xi})}\sigma+a\left(-\frac{i}{2}\right)\frac{1}{\sqrt{-i(\xi+\overline{\xi})}}\sigma+$$

$$
\begin{aligned}
& a\sqrt{-i(\xi+\overline{\xi})}\sigma\sigma\overline{a}\sigma \\
& = \sigma\overline{a}\frac{i}{2}\frac{1}{\sqrt{-i(\xi+\overline{\xi})}} + \\
& \sigma\overline{a}\sigma a\sqrt{-i(\xi+\overline{\xi})}\sigma.
\end{aligned} \tag{2.9}
$$

If $\lambda \to i\overline{\xi}$, then

$$
\begin{aligned}
& f_{\overline{\xi}} \sim -\frac{i}{2}\frac{1}{\sqrt{i(\xi+\overline{\xi})}}\frac{1}{\sqrt{\lambda-i\overline{\xi}}}, \\
& \left(\frac{1}{f}\right)_{\xi} \sim \frac{i}{2}\frac{1}{\sqrt{i(\xi+\overline{\xi})}}\frac{1}{\sqrt{\lambda-i\overline{\xi}}}, \\
& \frac{1}{f} \sim \sqrt{i(\xi+\overline{\xi})}\frac{1}{\sqrt{\lambda-i\overline{\xi}}}, \\
& a\left(-\frac{i}{2}\right)\frac{1}{\sqrt{i(\xi+\overline{\xi})}}\sigma + a\sigma\overline{a}\sqrt{i(\xi+\overline{\xi})} = \\
& \sigma\overline{a}_{\xi}\sqrt{i(\xi+\overline{\xi})} + \sigma\overline{a}\left(\frac{i}{2}\right)\frac{1}{\sqrt{i(\xi+\overline{\xi})}} + \sigma\overline{a}\sqrt{i(\xi+\overline{\xi})}a.
\end{aligned} \tag{2.10}
$$

If one substitutes the definitions of a, A, B: (2.6),(2.3) and (2.4) into (2.8), then a straightforward computation shows that (2.8) is an identity. The same substitution may be applied to (2.9) and (2.10). Equations (2.9) and (2.10) appear to be exactly (2.5). Thus, $\mathcal{E}$ solves (2.5) iff equations (2.1) and (2.2) are compatible.

CHAPTER 3

Riemann-Hilbert problem.

What follows is a presentation of a Riemann-Hilbert problem (RHP) which produces some solutions of the Lax couple [Korot].

Let $\mathcal{L}_0$ be the Riemann surface of the function $h(\lambda) = \sqrt{(\lambda + i\xi)(\lambda - i\overline{\xi})}$ and let $\Psi = \Psi(., \xi, \overline{\xi})$ be an analytic 2×2 invertible matrix function everywhere on $\mathcal{L}_0$, except for a finite number of points and cuts specified by conditions (1)-(3) below. The points $-i\xi$ and $i\overline{\xi}$ never coincide with these points and never lie on the cuts.

Assume that Ψ satisfies the following set of conditions.

1). Ψ is allowed to have jumps on the cuts of the form

$$\Psi_+ = \Psi_- v,$$

where v is a non singular constant 2×2 matrix. The position of the cuts is independent of ξ and $\overline{\xi}$.

2). Ψ is allowed to have singularities at a finite number of points. The position of these singularities is independent of ξ and $\overline{\xi}$. In terms of a local parameter s around the singularity, Ψ should have one of two possible factorizations near $s = 0$,

$$\Psi = \hat{\Psi}(s) s^T C$$

or

$$\Psi = \hat{\Psi}(s) e^{\frac{1}{s}T} C,$$

where $\hat{\Psi}(s)$ is an analytic invertible matrix function, T is a constant diagonal matrix and C is an invertible constant matrix.

3). $det\Psi$ is allowed to have a finite number of zeros. The location of these zeros should not depend on ξ and $\overline{\xi}$. The matrix Ψ should have a factorization of the form

$$\Psi = \hat{\Psi}(s)s^T C,$$

with the notation of condition (2).

4). Ψ should satisfy the symmetry condition

$$\Psi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.1}$$

where $*$ denotes involution of the sheets:

$$\Psi^*(\lambda) = \Psi(\lambda^*),$$

where λ^* has the same projection on a complex plane as λ, but is located on a different sheet of $\mathcal{L}_0$.

5). The condition at infinity is

$$\Psi(\infty^+) = \begin{pmatrix} \overline{\mathcal{E}} & 1 \\ \mathcal{E} & -1 \end{pmatrix}, \tag{3.2}$$

where ∞^+ is one of the infinities on $\mathcal{L}_0$, where $f(\infty^+) = 1$.

We now show, as in [Korot], that such a Ψ is a solution of the Lax couple of Chapter 2:

$$\Psi_\xi = \left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + f \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \right] \Psi,$$

$$\Psi_{\overline{\xi}} = \left[\begin{pmatrix} \overline{B} & 0 \\ 0 & \overline{A} \end{pmatrix} + \frac{1}{f}\begin{pmatrix} 0 & \overline{B} \\ \overline{A} & 0 \end{pmatrix}\right]\Psi,$$

$$\Psi(\infty^+) = \begin{pmatrix} \overline{\mathcal{E}} & 1 \\ \mathcal{E} & -1 \end{pmatrix},$$

$$A = \frac{\overline{\mathcal{E}}_\xi}{\overline{\mathcal{E}} + \mathcal{E}}, \tag{3.3}$$

$$B = \frac{\mathcal{E}_\xi}{\overline{\mathcal{E}} + \mathcal{E}}, \tag{3.4}$$

$$f = f(\lambda, \xi, \overline{\xi}) = \sqrt{\frac{\lambda - i\overline{\xi}}{\lambda + i\xi}}.$$

Let us observe first that conditions (3.1) and (3.2) imply that Ψ is bounded at both infinities. The conditions of the RHP are chosen in such a way that the combinations $\Psi_\xi \Psi^{-1}$ and $\Psi_{\overline{\xi}}\Psi^{-1}$ are analytic functions of λ everywhere on $\mathcal{L}_0$, except possibly for the points $-i\xi$ and $i\overline{\xi}$. For example, let us verify that $\Psi_\xi \Psi^{-1}$ is analytic at the zeros of $det\Psi$. Indeed, according to condition (3),

$$\Psi_\xi \Psi^{-1} = \left(\hat{\Psi}(s)s^T C\right)_\xi \left(\hat{\Psi}(s)s^T C\right)^{-1} = \hat{\Psi}_\xi(s)s^T C C^{-1} s^{-T}\hat{\Psi}^{-1}(s) = \hat{\Psi}_\xi(s)\hat{\Psi}^{-1}(s),$$

where $\hat{\Psi}$ is an analytic and invertible function around $s = 0$.

We next show that $\Psi_\xi \Psi^{-1}$ has a simple pole at $-i\xi$. Let s be a local parameter around $-i\xi$ so that

$$\lambda = -i\xi + s^2. \tag{3.5}$$

Then $\Psi = \Psi_0 + s\Psi_1 + O(s^2)$. Here λ is kept fixed when computing Ψ_ξ. It follows from (3.5) that $0 = -i + 2ss_\xi$, or $s_\xi = \frac{i}{2s}$. Therefore, $\Psi_\xi \Psi^{-1}$ has a simple pole at $-i\xi$. Similarly, $\Psi_{\overline{\xi}}\Psi^{-1}$ has a simple pole at $i\overline{\xi}$.

Let us introduce

$$f = \sqrt{\frac{\lambda - i\bar{\xi}}{\lambda + i\xi}}.$$

as a new variable. We have a one-to-one mapping $\lambda \to f$ of the surface $\mathcal{L}_0$ onto the complex plane. The point $-i\xi$ maps to infinity. The quantity $\Psi_\xi \Psi^{-1}$ as a function of f is analytic exept for infinity, where it has a simple pole. Therefore, $\Psi_\xi \Psi^{-1} = \mathcal{A} + f\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are constant matrices independent of f. A similar argument applies to $\Psi_{\bar{\xi}} \Psi^{-1}$.

We now use conditions (3.1) and (3.2) to establish precise expressions for $\mathcal{A}$ and $\mathcal{B}$. Write (3.1) in the form

$$\Psi = \epsilon \Psi^* \sigma,$$

where $\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as before, and take a derivative with respect to ξ:

$$\Psi_\xi = \epsilon \Psi^*_{\xi} \sigma,$$

Thus, we have two expressions for Ψ_ξ:

$$\Psi_\xi = [\mathcal{A} + f\mathcal{B}]\, \Psi,$$

and

$$\Psi_\xi = \epsilon \Psi^*_{\xi} \sigma.$$

Therefore,

$$\Psi_\xi = \epsilon \Psi^*_{\xi} \sigma =$$

$$\epsilon([\mathcal{A} + f\mathcal{B}]\,\Psi)^*\sigma =$$

$$\epsilon\,[\mathcal{A} + f^*\mathcal{B}]\,\Psi^*\sigma =$$

$$\epsilon\,[\mathcal{A} - f\mathcal{B}]\,\epsilon^{-1}\Psi\sigma^{-1}\sigma.$$

Hence,

$$[\mathcal{A} + f\mathcal{B}] = \epsilon\,[\mathcal{A} - f\mathcal{B}]\,\epsilon^{-1}$$

or

$$\mathcal{A} = \epsilon\mathcal{A}\epsilon^{-1}, \tag{3.6}$$

$$\mathcal{B} = -\epsilon\mathcal{B}\epsilon^{-1}. \tag{3.7}$$

Writing

$$\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$
$$\mathcal{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

then (3.6) implies that

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{pmatrix},$$

so $\mathcal{A}$ is of the form

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

Also

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} -B_{11} & B_{12} \\ B_{21} & -B_{22} \end{pmatrix},$$

so $\mathcal{B}$ is of the form

$$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

Consider the equation

$$\Psi_\xi = [\mathcal{A} + f\mathcal{B}]\,\Psi$$

at infinity. According to condition (3.2) we have

$$\begin{pmatrix} \overline{\mathcal{E}} & 1 \\ \mathcal{E} & -1 \end{pmatrix}_\xi \begin{pmatrix} \overline{\mathcal{E}} & 1 \\ \mathcal{E} & -1 \end{pmatrix}^{-1} = [\mathcal{A} + \mathcal{B}] = \begin{pmatrix} A_{11} & B_{12} \\ B_{21} & A_{22} \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} \overline{\mathcal{E}} & 1 \\ \mathcal{E} & -1 \end{pmatrix}_\xi \begin{pmatrix} \overline{\mathcal{E}} & 1 \\ \mathcal{E} & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\overline{\mathcal{E}}_\xi}{\mathcal{E}+\overline{\mathcal{E}}} & \frac{\overline{\mathcal{E}}_\xi}{\mathcal{E}+\overline{\mathcal{E}}} \\ \frac{\mathcal{E}_\xi}{\mathcal{E}+\overline{\mathcal{E}}} & \frac{\mathcal{E}_\xi}{\mathcal{E}+\overline{\mathcal{E}}} \end{pmatrix} = \begin{pmatrix} A_{11} & B_{12} \\ B_{21} & A_{22} \end{pmatrix}.$$

A similar computation applies to the second equation of the Lax couple

$$\Psi_{\overline{\xi}} = \left[\hat{\mathcal{A}} + \frac{1}{f}\hat{\mathcal{B}}\right]\Psi.$$

At infinity we have

$$\begin{pmatrix} \overline{\mathcal{E}} & 1 \\ \mathcal{E} & -1 \end{pmatrix}_{\overline{\xi}} \begin{pmatrix} \overline{\mathcal{E}} & 1 \\ \mathcal{E} & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\overline{\mathcal{E}}_{\overline{\xi}}}{\mathcal{E}+\overline{\mathcal{E}}} & \frac{\overline{\mathcal{E}}_{\overline{\xi}}}{\mathcal{E}+\overline{\mathcal{E}}} \\ \frac{\mathcal{E}_{\overline{\xi}}}{\mathcal{E}+\overline{\mathcal{E}}} & \frac{\mathcal{E}_{\overline{\xi}}}{\mathcal{E}+\overline{\mathcal{E}}} \end{pmatrix} = \begin{pmatrix} \hat{A}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{A}_{22} \end{pmatrix}.$$

Theorem. The solution of the RHP (1)-(5) is unique.

Proof. The key point here is that the function $\mathcal{E}$ in condition (5) is “some ”. In condition (5) we require only that the asymptotics of the solution at infinity is of a specified form. We are assuming that there are two solutions Ψ_1 and Ψ_2

satisfying condition (5) with two functions $\mathcal{E}_1$ and $\mathcal{E}_2$ and all other parameters of the RHP are the same. We proceed to prove that the functions Ψ_1 and Ψ_2 are identical.

Observe that the RHP is such that the combination $\Psi_1\Psi_2^{-1}$ is a bounded analytic function on the Riemann surface $\mathcal{L}_0$. Therefore, $\Psi_1\Psi_2^{-1} = A$, where A is a matrix, possibly dependent on the variables ξ and $\overline{\xi}$ and independent of the variable λ.

Using symmetry condition (4), we have

$$\Psi_1 = A\Psi_2,$$

$$\epsilon\Psi_2\sigma = \Psi_2^*,$$

$$\epsilon A\Psi_2\sigma = A\Psi_2^*.$$

Therefore,

$$A\epsilon = \epsilon A. \tag{3.8}$$

We conclude that A is a diagonal matrix using a similar argument.

Using the condition (5),

$$\Psi_1(\infty^+) = \begin{pmatrix} \overline{\mathcal{E}_1} & 1 \\ \mathcal{E}_1 & -1 \end{pmatrix}$$

and

$$\Psi_2(\infty^+) = \begin{pmatrix} \overline{\mathcal{E}_2} & 1 \\ \mathcal{E}_2 & -1 \end{pmatrix}.$$

Writing

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix},$$

we have

$$\begin{pmatrix} \overline{\mathcal{E}_1} & 1 \\ \mathcal{E}_1 & -1 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} \overline{\mathcal{E}_2} & 1 \\ \mathcal{E}_2 & -1 \end{pmatrix}.$$

Comparing entries 12 and 22, we conclude that $a_1 = a_2 = 1$.

CHAPTER 4

Solution of the Riemann-Hilbert problem.

In this chapter we produce a solution of the RHP (1)-(5) discussed in Chapter 3. The construction presented here is a slight modification of the construction in [Korot]. By the result of the Chapter 3, it follows that we have constructed a class of solutions of EVE.

Let us consider the consequences of the symmetry condition

$$\Psi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.1}$$

We have

$$\begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} = \begin{pmatrix} \Psi_{12}^* & \Psi_{11}^* \\ -\Psi_{22}^* & -\Psi_{21}^* \end{pmatrix}.$$

Hence, the solution Ψ has to be of the form

$$\Psi = \begin{pmatrix} \Psi_1 & \Psi_1^* \\ \Psi_2 & -\Psi_2^* \end{pmatrix},$$

where $\Psi_1 = \Psi_{11}$, $\Psi_2 = \Psi_{21}$.

Observe that all conditions of the RHP, except for the condition at infinity

$$\Psi(\infty^+) = \begin{pmatrix} \overline{\mathcal{E}} & 1 \\ \mathcal{E} & -1 \end{pmatrix}$$

are invariant under transformation $\Psi \mapsto C\Psi$, where C is a constant diagonal matrix. If C is chosen so that

$$C = \begin{pmatrix} \frac{1}{\Psi_1^*(\infty^+)} & 0 \\ 0 & \frac{1}{\Psi_2^*(\infty^+)} \end{pmatrix}, \tag{4.2}$$

then the transformation $\Psi \mapsto C\Psi$ produces a solution with the property

$$\Psi(\infty^+) = \begin{pmatrix} \mathcal{E}_1 & 1 \\ \mathcal{E}_2 & -1 \end{pmatrix},$$

for some $\mathcal{E}_1$ and $\mathcal{E}_2$. Introduce the notation $\tilde{}: f(\lambda) \mapsto \tilde{f}(\lambda) = \overline{f(\overline{\lambda})}$. We want $\mathcal{E}_1 = \overline{\mathcal{E}_2}$ and this motivates us to look for solutions of the form $\Psi_1 = \tilde{\Psi}_2 = \chi$ and

$$\Psi = \begin{pmatrix} \chi & \chi^* \\ \tilde{\chi} & -\tilde{\chi}^* \end{pmatrix}. \tag{4.3}$$

Let $\mathcal{L}$ be the Riemann surface of the function $h(\lambda) = \sqrt{(\lambda + i\xi)(\lambda - i\overline{\xi}) \prod_{i=1}^{g} (\lambda - a_i)(\lambda - b_i)}$, where a_i and b_i are real and do not depend on ξ and $\overline{\xi}$. The surface is represented as two copies (sheets) of the complex plane with the cuts $[a_i, b_i]$ and $[-i\xi, i\overline{\xi}]$ (Figure 4.1.).

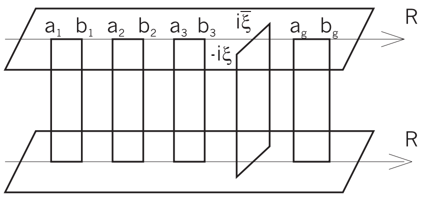

Figure 4.1: The surface $\mathcal{L}$.

The surface is glued across the cuts in the standard way (i.e., for horizontal cuts this means that the upper(lower) side of the cut on the upper sheet is glued with the lower(upper) side of the cut on the lower sheet). We call the cuts $[a_i, b_i]$

"non moving cuts ", since their position does not depend on parameters ξ and $\overline{\xi}$. The cut $[-i\xi, i\overline{\xi}]$ is called "the moving cut ". The orientation of $\mathcal{A}$ and $\mathcal{B}$ periods is shown on Figure 4.2, for the case $g = 4$.

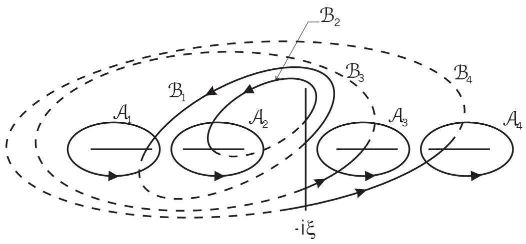

Figure 4.2: Orientation of $\mathcal{A}$ and $\mathcal{B}$ periods.

The dotted line denotes a part of the path located on the second sheet of $\mathcal{L}$. Let $\{u_i\}_{i=1}^g$ be a basis of holomorphic differentials on $\mathcal{L}$ normalized by the condition $\int_{\mathcal{A}_i} u_j = \delta_{ij}$. Introduce the multi-valued function $U_i(\lambda) = \int_{i\xi}^{\lambda} u_i$. We look for χ of the form

$$\chi(\lambda) = C\frac{\theta(U(\lambda) - U(D) - K + \frac{B_w}{2\pi i} + \Pi)}{\theta(U(\lambda) - U(D) - K)}e^W. \tag{4.4}$$

Here, C is a normalization constant, $U = \{U_i\}_{i=1}^g$, $D = \{D_1, ..., D_g\}$ is a non-special divisor, $U(D) = \sum_{i=1}^g U(D_i)$, $K = \{K_i\}_{i=1}^g$, $K_j = \frac{1}{2}\sum_{i=1}^g b_{ij} - \frac{j}{2}$, $W(\lambda) = \int_{i\xi}^{\lambda} w$ is an abelian integral of the second kind, normalized by the conditions $\int_{\mathcal{A}_i} w = 0$, $B_w = \{(B_w)_i\}_{i=1}^g$, $(B_w)_i = \int_{\mathcal{B}_i} w$, $\Pi = \{r, r, ..., r\}$ is a vector of equal rational entries r and $\theta(z) = \sum_{m \in Z^g} exp\{\pi i < Bm, m > +2\pi i < z, m >\}$, $B = (b_{ij})_{j=1,...,g}^{i=1,...,g}$, $b_{ij} = \int_{\mathcal{B}_i} u_j$ is the theta function. The abelian integral $W(\lambda)$ is given explicitly in (4.13).

We extend the notion of the involution defined in the previous chapter to multivalued functions. As before, if λ is a point on one sheet of $\mathcal{L}$, then λ^* is the point on another sheet with the same projection down to the complex plane.

Recall that if $f(\cdot)$ is a single valued function on $\mathcal{L}$, then f^* is defined according to $f^*(\lambda) = f(\lambda^*)$. Suppose $f(\cdot)$ is not single valued on $\mathcal{L}$ and the value $f(P)$ is given by the continuation through the path shown on Figure 4.3.

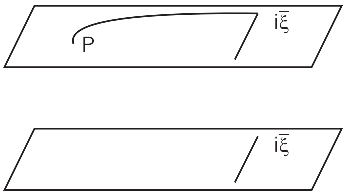

Figure 4.3: Sample path to a point λ.

Then by definition, the value $f^*(P)$ is given by the path shown on Figure 4.4. The two paths have the same projections in the complex plane.

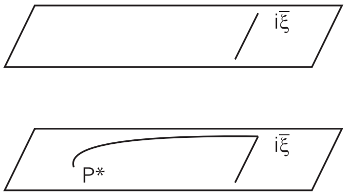

Figure 4.4: Sample path to a point λ^*.

If λ is located on the first sheet, away from the cuts, and if we choose a path of integration entirely on the first sheet and also away from any of the cuts, then the function χ is single valued. Let's call this branch a principal branch and denote it by the same symbol χ. We have the properties $U^* = -U$ and $W^* = -W$. One observes that if χ is given by (4.4) then χ^* is given by

$$\chi^*(\lambda) = C\frac{\theta(-U(\lambda) - U(D) - K + \frac{B_w}{2\pi i} + \Pi)}{\theta(-U(\lambda) - U(D) - K)}e^{-W}.$$

We need to study the jumps of χ on the cuts, so let us consider the jumps of U. We introduce the notation $U_\pm(x) = \lim_{\epsilon\downarrow 0} U(x \pm i\epsilon)$ for horizontal cuts and

$U_\pm(ix) = \lim_{\epsilon\downarrow 0} U(\pm\epsilon + ix)$ for vertical cuts. Figure 4.5 shows paths of integration whose sum is $U_+ + U_-$.

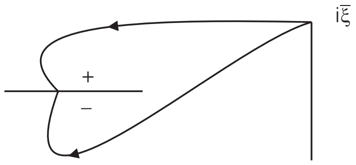

Figure 4.5: Paths of integration giving U_+ and U_-.

Integrating around the paths as in Figure 4.6

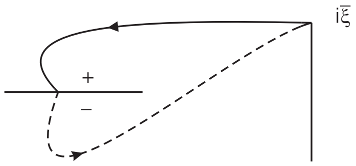

Figure 4.6: Paths of integration giving U_+ and U_-.

we obtain the relation

$$U_+ + U_- = B_k + \sum A_i$$

on the k-th non-moving cut, where $\sum A_i$ denotes an appropriate sum of the $\mathcal{A}$ periods of U and B_k is the k-th column of the matrix B.

The presence of the $\mathcal{A}$ periods is due to the fact that there are other cuts in the plane, so the path may go in many different ways as shown on Figure 4.7.

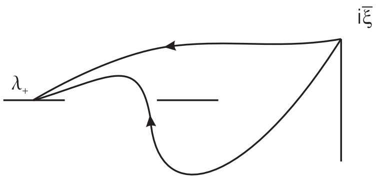

Figure 4.7: Two different ways to reach λ_+.

Since U is used as a part of the argument of the θ function, the precise number of $\mathcal{A}$ periods in the sum is irrelevant (see below). On the moving cut we have

$$U_+ + U_- = 0 + \sum A_i.$$

A similar computation holds for the abelian integral of the second kind W.

We are now in a position to compute the jumps of χ using the following properties of the θ function [FarKra]:

$$\theta(s \pm B_j) = e^{\mp 2\pi i s_j - \pi i B_{jj}}\theta(s)$$

$$\theta(s \pm e_j) = \theta(s),$$

where B_j is the j-th column of the matrix B and e_j is a j-th column of the unit matrix. We have

$$\begin{aligned}
\chi_+ &= C\frac{\theta(U_+ - U(D) - K + \frac{B_w}{2\pi i} + \Pi)}{\theta(U_+ - U(D) - K)}e^{W_+} \\
&= C\frac{\theta(-U_- + B_k - U(D) - K + \frac{B_w}{2\pi i} + \Pi)}{\theta(-U_- + B_k - U(D) - K)}e^{-W_- + B_{wk}} \\
&= C\frac{e^{-2\pi i(-U_- - U(D) - K + \frac{B_w}{2\pi i} + \Pi)_k - \pi i B_{kk}}\theta(-U_- - U(D) - K + \frac{B_w}{2\pi i} + \Pi)}{e^{-2\pi i(-U_- - U(D) - K)_k - \pi i B_{kk}}\theta(-U_- - U(D) - K)}e^{-W_- + B_{wk}} \\
&= C\frac{\theta(-U_- - U(D) - K + \frac{B_w}{2\pi i} + \Pi)}{\theta(-U_- - U(D) - K)}e^{-W_-}e^{-2\pi i \Pi_k} \\
&= \chi_-^* e^{-2\pi i \Pi_k},
\end{aligned} \tag{4.5}$$

i.e.

$$\chi_+ = \chi_-^* e^{-2\pi i \Pi_k}. \tag{4.6}$$

On the moving cut,

$$\chi_+ = C\frac{\theta(U_+ - U(D) - K + \frac{B_w}{2\pi i} + \Pi)}{\theta(U_+ - U(D) - K)}e^{W_+}$$
$$= C\frac{\theta(-U_- - U(D) - K + \frac{B_w}{2\pi i} + \Pi)}{\theta(-U_- - U(D) - K)}e^{-W_-},$$

or

$$\chi_+ = \chi_-^*. \tag{4.7}$$

The difference in jump conditions across the non-moving and moving cuts indicates that we are obtaining different branches of χ when continuing χ through moving and non-moving cuts: Figures 4.8 and 4.9. We see from (4.7) that χ^* is obtained by continuing χ through the moving cut. However, from (4.6), this is not true for continuing through a non-moving cut.

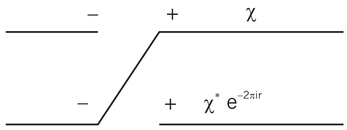

Figure 4.8: Non moving cut gluing.

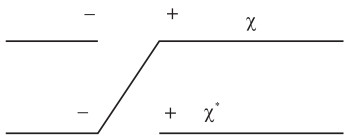

Figure 4.9: Moving cut gluing.

A similar computation for χ_- gives

$$\chi_- = C\frac{\theta(U_- - U(D) - K + \frac{B_w}{2\pi i} + \Pi)}{\theta(U_- - U(D) - K)}e^{W_-}$$
$$= C\frac{\theta(-U_+ + B_k - U(D) - K + \frac{B_w}{2\pi i} + \Pi)}{\theta(-U_+ + B_k - U(D) - K)}e^{-W_+ + B_{wk}}$$

$$
\begin{aligned}
&= C\frac{e^{-2\pi i(-U_+-U(D)-K+\frac{B_w}{2\pi i}+\Pi)_k-\pi i B_{kk}}\theta(-U_+-U(D)-K+\frac{B_w}{2\pi i}+\Pi)}{e^{-2\pi i(-U_+-U(D)-K)_k-\pi i B_{kk}}\theta(-U_+-U(D)-K)}e^{-W_++B_{wk}}\\
&= C\frac{\theta(-U_+-U(D)-K+\frac{B_w}{2\pi i}+\Pi)}{\theta(-U_+-U(D)-K)}e^{-W_+}e^{-2\pi i\Pi_k}\\
&= \chi_+^* e^{-2\pi i \Pi_k} \qquad (4.8)
\end{aligned}
$$

on the non-moving cut and

$$\chi_- = \chi_+^* \qquad (4.9)$$

on the moving cut.

We focus our attention on the surface $\mathcal{L}_0$, which consists of two copies of $C\setminus\bigcup_{i=1}^{g}[a_i,b_i]$ glued in the standard way across the moving cut $[-i\xi, i\overline{\xi}]$.

The principal branch χ is located on the first(or upper) sheet of the surface. By (4.7) and (4.9), continuation through the moving cut to the second sheet is given by the branch χ^*.

We now set $r = -\frac{1}{4}$ and verify that the matrix function (4.3) with such a χ is a solution of the Riemann-Hilbert problem with the desired properties listed in Chapter 3. To this point

$$\Psi = \begin{pmatrix} \chi & \chi^* \\ \tilde{\chi} & -\tilde{\chi}^* \end{pmatrix},$$

and by (4.6) and (4.8) with $r = -\frac{1}{4}$,

$$\chi_+ = i\chi_-^*,$$

$$\chi_- = i\chi_+^*$$

on the non moving cuts and

$$\chi_+ = \chi_-^*,$$

$$\chi_- = \chi_+^*$$

on the moving cut.

Set $\eta = \tilde{\chi}$. Then across horizontal cuts $\overline{\lambda}_+ = \lambda_-$

$$\eta_+ = \overline{\chi(\overline{\lambda_+})} = \overline{\chi(\lambda_-)} = \overline{i\chi^*(\lambda_+)} = \overline{i\chi^*(\overline{\lambda_-})} = (-i)\overline{\chi^*(\overline{\lambda_-})} = (-i)\eta_-^*.$$

Similarly,

$$\eta_- = \overline{\chi(\overline{\lambda_-})} = \overline{\chi(\lambda_+)} = \overline{i\chi^*(\lambda_-)} = \overline{i\chi^*(\overline{\lambda_+})} = (-i)\overline{\chi^*(\overline{\lambda_+})} = (-i)\eta_+^*.$$

Therefore,

$$\Psi_+ = \begin{pmatrix} \chi_+ & \chi_+^* \\ \eta_+ & -\eta_+^* \end{pmatrix} = \begin{pmatrix} i\chi_-^* & -i\chi_- \\ -i\eta_-^* & -i\eta_- \end{pmatrix} =$$

$$i\begin{pmatrix} -\chi_- & \chi_-^* \\ -\eta_- & -\eta_-^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\begin{pmatrix} \chi_- & \chi_-^* \\ \eta_- & -\eta_-^* \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Psi_- i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus

$$\Psi_+ = \Psi_- v, \tag{4.10}$$

where the jump matrix v is given by

$$v = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.11}$$

Notice that $det\, v = -1$. On the moving cut,

$$\Psi_+ = \begin{pmatrix} \chi_+ & \chi_+^* \\ \eta_+ & -\eta_+^* \end{pmatrix} = \begin{pmatrix} \chi_-^* & \chi_- \\ \eta_-^* & -\eta_- \end{pmatrix} =$$

$$\begin{pmatrix} \chi_- & \chi_-^* \\ -\eta_- & \eta_-^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \chi_- & \chi_-^* \\ \eta_- & -\eta_-^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi_- \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

in agreement with the required symmetry condition (4.1).

We now locate the zeros of $det\Psi$. We previously have established that $det\Psi$ satisfies the jump conditions

$$det\Psi_+ = -det\Psi_- \tag{4.12}$$

on the non-moving cuts and by (4.1) it does not jump on the moving cut. The parameters of W will be chosen later (see 4.13) in such a way that $W^* = -W$ and also $\tilde{W} = W$, so e^W drops out from the expression for $det\Psi$. Then the only singularities of $det\Psi$ are given by the non-special divisors D and D^*. Thus, $det\Psi$ has g zeros on both of the two planes of $\mathcal{L}_0$. Introduce the function

$$R(\lambda) = p(\lambda) det\Psi$$

on the upper sheet of $\mathcal{L}_0$, where $p(\lambda) = \sqrt{\prod_{i=1}^{g}(\lambda - a_i)(\lambda - b_i)}$ and the branch is chosen so that $p \to \lambda^g$ as $\lambda \to \infty$. Using (4.12) we see that $R(\lambda)$ is a single valued meromorphic function on the upper sheet of $\mathcal{L}_0$. It follows from the asymptotic condition (3.2) for Ψ that $det\Psi$ is bounded at infinity, and hence, $R(\lambda)$ grows like

$const.\lambda^g$ as $\lambda \to \infty$. Therefore, $R(\lambda)$ has g poles and $2g$ zeros in the complex plane. Recall that the divisors D and D^* are non-special and hence none of the branch points a_i, b_i, $i = 1, ..., g$ can be in D or D^*. Hence, $det\Psi$ is bounded around the points a_i, b_i. Now $p(\lambda)$ vanishes like $\lambda^{\frac{1}{2}}$ at every non-moving branch point, but at the same time, $R(\lambda)$ is meromorphic. Therefore, $det\Psi$ must vanish at every non-moving branch point, at least like $\lambda^{\frac{1}{2}}$. So, the $2g$ branch points give at least $2g$ zeros of R. Therefore, these are all the zeros of R and hence all points where $det\Psi$ vanishes. We also conclude that $det\Psi$ always vanishes precisely like $(\lambda - const)^{\frac{1}{2}}$ in the complex plane.

Next, we establish a factorization condition for Ψ at zeros of $det\Psi$. Introduce

$$N = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix},$$

where

$$p = \sqrt{\prod_{k=1}^{g} (\lambda - a_i)(\lambda - b_i)}$$

and p has the property $p_+ = -p_-$ across the non-moving cuts. Therefore,

$$N_+ = \begin{pmatrix} 1 & 0 \\ 0 & p_+ \end{pmatrix} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -p_- \end{pmatrix} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & p_- \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p_- \end{pmatrix} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & p_- \end{pmatrix} i \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p_- \end{pmatrix} i \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & p_- \end{pmatrix} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = N_- i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which is precisely the same jump condition as (4.10)-(4.11) established for Ψ. If we write $\Psi = AN$, then A has to be an analytic matrix function around the cut. Furthemore, we know from above that $det\Psi$ vanishes exactly as $(\lambda - const)^{\frac{1}{2}}$ on the complex plane near each of the non-moving branch points. Clearly, these are also points where $detN$ vanishes like $(\lambda - const)^{\frac{1}{2}}$. Thus, A is invertible near all of the non-moving branch points.

Theorem. Suppose that D satisfies the symmetry condition $D = \overline{D}$. The branches Ψ and Ψ^* constitute a solution of the RHP.

Proof. It remains to check the factorization conditions at the singularities of Ψ. There are poles arising from D and D^* and there are singular points arising from poles of W in e^W. Let λ_0 be a pole arising from the divisor D and let s be a local parameter around λ_0. We have

$$\Psi = \begin{pmatrix} \chi & \chi^* \\ \eta & -\eta^* \end{pmatrix} = \begin{pmatrix} \frac{1}{s}\hat{\chi} & \chi^* \\ \frac{1}{s}\hat{\eta} & -\eta^* \end{pmatrix} = \begin{pmatrix} \hat{\chi} & \chi^* \\ \hat{\eta} & -\eta^* \end{pmatrix} s^{\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}}$$

for some functions $\hat{\chi}$ and $\hat{\eta}$ which are analytic around λ_0. Hence, Ψ has the desired form.

We investigate the singularities of W, that is given by the expressions

$$W = \sum_{k=1}^{g} \alpha_k \left(W(p_k, \lambda) - W(p_k^*, \lambda) \right), \tag{4.13}$$

$$W(p, \lambda) = \int_{i\xi}^{\lambda} \frac{h(s) + h(p) + h'(p)(s-p)}{(s-p)^2 h(s)} ds,$$

$$W(p^*, \lambda) = \int_{i\xi}^{\lambda} \frac{h(s) - h(p) - h'(p)(s-p)}{(s-p)^2 h(s)} ds,$$

$$h(s) = \sqrt{(s+i\xi)(s-i\overline{\xi})\prod_{i=1}^{g}(s-a_i)(s-b_i)},$$

where p_k and α_k are real numbers. Observe that $W^*(p,\lambda) = W(p^*,\lambda)$ and $W^* = -W$. According to the above relations,

$$W = \tilde{W} = \frac{\alpha_k}{s}(1+O(1)) \tag{4.14}$$

$$W^* = \tilde{W}^* = -\frac{\alpha_k}{s}(1+O(1)) \tag{4.15}$$

for a local parameter s around $\lambda = p_k$, for some k. Thus, Ψ has the following decomposition around the singularities of W,

$$\Psi = \hat{\Psi}\begin{pmatrix} e^{\frac{\alpha}{s}} & 0 \\ 0 & e^{-\frac{\alpha}{s}} \end{pmatrix} = \hat{\Psi} e^{\frac{\alpha}{s}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}},$$

which is again of the desired form.

This concludes the proof that the function Φ that we have constructed in (4.3),(4.4) is a solution of the RHP (1)-(5) of the desired type.

CHAPTER 5

Some classical solutions.

In this chapter, for the reader's convenience, we present some well known solutions of the EVE's.

The space-time of a non-rotating black hole is given by the Schwartchild metric:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) + \left(1 - \frac{2m}{r}\right)^{-1} dr^2. \tag{5.1}$$

We compute expressions for Weyl's canonical coordinates (ρ, z) (see Chapter 1). According to (1.2),

$$\rho^2 = -VX + W^2,$$

where V, X and W are quantities coming from the general expression (1.1):

$$ds^2 = Vdt^2 + 2Wdtd\phi + Xd\phi^2 + g_{AB}dx^A dx^B. \tag{5.2}$$

Comparing (5.1) and (5.2),

$$V = -\left(1 - \frac{2m}{r}\right),$$

$$W = 0$$

and

$$X = r^2 \sin^2\theta.$$

Therefore,

$$\rho^2 = \left(1 - \frac{2m}{r}\right) r^2 \sin^2\theta.$$

We choose $\rho > 0$, and consider space time outside of the event horizon, so

$$\rho = r\left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} \sin\theta,\ 0 \le \theta \le \pi,\ r > 2m.$$

Let's compute the coordinate z: z-lines are defined to be orthogonal to ρ-lines with respect to the given metric. So, let's take z according to

$$\partial_\alpha z = \epsilon_{\alpha\beta}\partial^\beta \rho,$$

where indices run through the values r and θ and ϵ is the completely antisymmetric 2-tensor, $\epsilon_{12} = \sqrt{g}$. For such z,

$$< \nabla z, \nabla\rho >= g^{\alpha,\beta}\partial_\alpha z \partial_\beta \rho = g^{\alpha,\beta}\epsilon_{\alpha\gamma}\partial_\gamma\rho\partial_\beta\rho = 0.$$

In this case,

$$g_{\alpha\beta} = r^2\sin^2\theta \begin{pmatrix} \left(1 - \frac{2m}{r}\right)^{-1} & 0 \\ 0 & r^2 \end{pmatrix},$$

$$g^{\alpha\beta} = \frac{1}{r^2\sin^2\theta} \begin{pmatrix} \left(1 - \frac{2m}{r}\right) & 0 \\ 0 & r^{-2} \end{pmatrix}$$

and

$$\sqrt{g} = \left(r^2\sin^2\theta\right)\left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} r.$$

Thus,

$$\partial_r z = \epsilon_{r\theta} g^{\theta\theta}\partial_\theta\rho =$$

$$\left(r^2\sin^2\theta\right)\left(1-\frac{2m}{r}\right)^{-\frac{1}{2}} r\left(r^2\sin^2\theta\right)^{-1} r^{-2} r\left(1-\frac{2m}{r}\right)^{\frac{1}{2}} Cos\theta =$$

$$= Cos\theta.$$

and

$$\partial_\theta z = (-1)\left(r^2\sin^2\theta\right)\left(1-\frac{2m}{r}\right)^{\frac{1}{2}} r\frac{1}{r^2\sin^2\theta}\left(1-\frac{2m}{r}\right)\frac{d}{dr}\left(r\left(1-\frac{2m}{r}\right)^{1/2}\sin\theta\right) =$$

$$-\sin\theta\,(r-m)\,.$$

Therefore,

$$z = (r-m)Cos\theta.$$

Note, that in $\{\rho, z\}$-space, level curves of r are ellipses. The event horizon is a degenerate ellipse $\{\rho = 0, z \in [-m, m]\}$.

According to formula (1.12),

$$ds^2 = e^{-2U}(e^{2K}(d\rho^2 + dz^2) + \rho^2 d\phi^2) - e^{2U}(dt + Ad\phi)^2,$$

$e^{2U} = Re\mathcal{E}$ is the negative of the coefficient in front of dt^2. Thus, in the Schwartchild case,

$$e^{2U} = \left(1-\frac{2m}{r}\right).$$

Notice, that

$$\rho^2 + (z+m)^2 = (-m + r + mCos\theta)^2$$

and

$$\rho^2 + (z-m)^2 = (m - r + mCos\theta)^2.$$

Outside of the event horizon,

$$\sqrt{\rho^2 + (z+m)^2} = -m + r + mCos\theta,$$

$$\sqrt{\rho^2 + (z-m)^2} = -(m - r + mCos\theta)$$

and

$$\sqrt{\rho^2 + (z+m)^2} + \sqrt{\rho^2 + (z-m)^2} = 2(r-m).$$

Thus,

$$e^{2U} = \left(1 - \frac{2m}{r}\right) =$$

$$\left(1 - \frac{4m}{\sqrt{\rho^2 + (z+m)^2} + \sqrt{\rho^2 + (z-m)^2} + 2m}\right) =$$

$$\frac{\sqrt{\rho^2 + (z+m)^2} + \sqrt{\rho^2 + (z-m)^2} - 2m}{\sqrt{\rho^2 + (z+m)^2} + \sqrt{\rho^2 + (z-m)^2} + 2m}.$$

We now introduce the so called prolate spheroidal coordinates $\{X, Y\}$ according to the following relations:

$$X = \frac{1}{2\sigma}(r_+ + r_-),$$

$$Y = \frac{1}{2\sigma}(r_+ - r_-),$$

$$r_\pm^2 = \rho^2 + (z \pm \sigma)^2,$$

where σ is some constant. The inverse relations are given by

$$\rho^2 = \sigma^2(X^2 - 1)(1 - Y^2),$$

$$z = \sigma XY.$$

We have

$$e^{2U} = \frac{X-1}{X+1},$$

with $\sigma = m$.

As one can see from relations (1.4)-(1.9), the imaginary part Φ of the Ernst potential $\mathcal{E}$ is determined up to a constant. Furthemore, the quantity A, which measures the rotation, is zero in the Schwartzchild case. Therefore, according to (1.6)-(1.7), Φ is a constant which we choose to be zero. Therefore, the Ernst potential for Schwartchild space time is given by

$$\mathcal{E} = \frac{X-1}{X+1}. \tag{5.3}$$

Notice that in the absence of rotation (Ernst potential is real) then the change of the unknown function $\mathcal{E} \to U$, $\mathcal{E} = e^{2U}$ makes the Ernst equation linear. Hence, one should expect that the space time of two non-rotating black holes (Weyl solution [BachWeyl]) is given by

$$e^{2U}(\rho, z) = \mathcal{E}(\rho, z) = \mathcal{E}_{Schw}(\rho, z)\mathcal{E}_{Schw}(\rho, z - R),$$

where $\mathcal{E}_{Schw}(\rho, z)$ is the Ernst potential of the Schwartzchild solution. Using the original article's notation we present the space time functions U and K for the Weyl's solution:

$$e^{2U} = \frac{(-2\,l + r_1 + r_2)\,(-2\,l' + r_3 + r_4)}{(2\,l + r_1 + r_2)\,(2\,l' + r_3 + r_4)},$$

$$e^{2K} = \left(d\,(d+l')\,r_1 + d\,(d+l+l')\,r_2 - d\,l\,r_4\right)^2 \times$$
$$\frac{\left(-4\,l^2 + (r_1+r_2)^2\right)\,\left(-4\,l'^2 + (r_3+r_4)^2\right)}{16\,r_1\,r_2\,r_3\,\left(d\,(d+l')\,r_1 + (d+l)\,(d+l')\,r_2 - l\,(d+l')\,r_3\right)^2 r_4},$$
$$l = m_1, l' = m_2, d = \frac{R}{2} - \sigma.$$

Here m_i, $i = 1, 2$ are masses of the bodies, the event horizons are located on the z-line at the intervals $[-m_1, m_1]$, $[R-m_2, R+m_2]$ and r_i, $i = 1, ..., 4$ are notations for $\sqrt{\rho^2 + (z-\lambda_i)^2}$, $\{\lambda_i\} = \{-m_1, m_1, R-m_2, R+m_2\}$.

Contrary to the Schwartzchild case, there are no uniformly agreed upon coordinates for the Kerr solution, and hence there is no uniformly agreed upon form for the metric ds^2. According to [KraSte], space time of a rotating black hole (Kerr solution) is given by

$$\mathcal{E} = \frac{pX - iqY - 1}{pX - iqY + 1}, \tag{5.4}$$

$$mq = a, mp = \sigma, p^2 + q^2 = 1,$$
$$e^{2U} = \frac{p^2X^2 + q^2Y^2 - 1}{(pX+1)^2 + q^2Y^2},$$
$$e^{2K} = \frac{p^2X^2 + q^2Y^2 - 1}{p^2(X^2 - Y^2)},$$
$$A = \frac{2mq}{p^2X^2 + q^2Y^2 - 1}(1 - Y^2)(pX + 1).$$

We regard these expressions as our definition of the Kerr solution. Observe that in these variables, the Kerr potential is equal to the Schwartzchild potential if the parameters p and q are chosen to be $p = 1$, $q = 0$

CHAPTER 6

Degenerate case.

In the notation of Chapter 3, let us consider a limit $a_i - b_i \to 0$, $a_i \to \lambda_i$, λ_i are real, for $i = 1, ..., g$.

What happens to the expression $\theta(U(\infty^+) + d + \frac{B}{2\pi i} + \Pi)$ under this limiting procedure? We have

$$d = -U(D) - K,$$

$$K_m = \frac{1}{2}\sum_{j=1}^{g} b_{mj} - \frac{m}{2},$$

$$D = \{\mu_1, ..., \mu_g\},$$

$$U(D) = \sum_{k=1}^{g} U(\mu_k),$$

$$\Pi = -\frac{1}{4},$$

$$U_s(\lambda) = -\frac{1}{2\pi i} log\beta(\lambda_s, \lambda),$$

$$\theta(s) = \sum_{k\in Z^g} exp\{2\pi i(s,k) + \pi i(bk,k)\},$$

$$b_{sj} = \int_{\mathcal{B}_s} u_j = \frac{-1}{\pi i} log\beta_{sj}.$$

We proceed with the computation of $\beta(\lambda_s, \lambda)$. According to the normalization condition $\int_{\mathcal{A}_f} u_k = \delta_{sk}$, when a cut $[a_i, b_i]$ shrinks into a point λ_i the point becames a residue of u_i. Furthemore, this point is the only singularity of u_i on

$\mathcal{L}$ which is othewise analytic and bounded. Therefore,

$$u_i = \frac{1}{2\pi i}\frac{h(\lambda_i)}{(\lambda-\lambda_i)h(\lambda)}d\lambda.$$

One can verify that

$$\int \frac{1}{2\pi i}\frac{h(a)}{(\lambda-a)h(\lambda)}d\lambda = \frac{(-1)}{2\pi i} log\alpha(a,\lambda),$$

$$\alpha(a,\lambda) = \frac{(h(a)+h(\lambda))^2-(a-\lambda)^2}{h(a)(\lambda-a)},$$

$$h(\lambda) = \sqrt{(\lambda+i\xi)(\lambda-i\overline{\xi})}.$$

If the path of integration in U is away from the cuts, and does not wind around any of them, then

$$U_k(\lambda) = \frac{(-1)}{2\pi i} log\frac{\alpha(\lambda_k,\lambda)}{\alpha(\lambda_k, i\overline{\xi})}.$$

Therefore,

$$\beta(a,\lambda) = \frac{(h(a)+h(\lambda))^2-(a-\lambda)^2}{(\lambda-a)}\frac{1}{(-i)(\xi+\overline{\xi})}. \tag{6.1}$$

Let us compute β_{sj}. The path $\mathcal{B}_s$ is shown on the Figure 6.1.

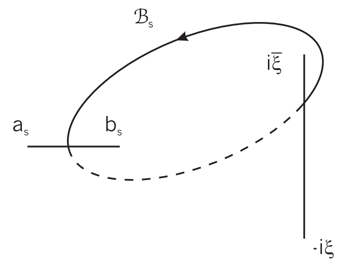

Figure 6.1: $\mathcal{B}_s$.

Under the limiting procedure, $[a_s, b_s]$ shrinks to a point λ_s (Figure 6.2).

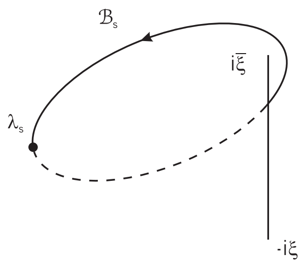

Figure 6.2: $\mathcal{B}_s$ on the degenerated surface.

We may want to shrink $\mathcal{B}_s$ into a path shown in Figure 6.3.

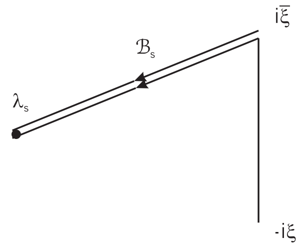

Figure 6.3: Shrinked $\mathcal{B}_s$.

Such a transformation is possible if $[a_j, b_j]$ lies outside of the domain bounded by $\mathcal{B}_s$. If this is so, then $\beta_{sj} = \beta(\lambda_j, \lambda_s)$. If this is not so, then we use symmetry of b_{sj} to reverse indices.

A straightforward computation shows that $Re(ib_{mm}) = -\infty$ under the limiting procedure.

According to the above definitions,

$$\theta(U(\infty^+) + d + \frac{B}{2\pi i} + \Pi) =$$

$$\sum_{k \in Z^g} exp\left\{2\pi i \sum_{m=1}^{g} \left(U(\infty^+) + d + \frac{B_w}{2\pi i} + \Pi\right)_m k_m + \pi i \sum_{m,l=1}^{g} b_{ml} k_l k_m\right\} = \ldots$$

$$d_m = (-U(D) - K)_m = -\sum_{k=1}^{g} U_m(\mu_k) - \frac{1}{2}\sum_{j=1}^{g} b_{mj} + \frac{m}{2},$$

$$\ldots = \sum_{k\in Z^g} exp\{2\pi i \sum_{m=1}^{g} \left(U(\infty^+) + \frac{B_w}{2\pi i} + \Pi - U(D)\right)_m k_m$$

$$+2\pi i \sum_{m=1}^{g} (-\frac{1}{2}\sum_{j=1}^{g} b_{mj} + \frac{m}{2})k_m + \pi i \sum_{m,l=1}^{g} b_{ml}k_l k_m\} =$$

$$= \sum_{k\in Z^g} exp\{2\pi i \sum_{m=1}^{g} \left(U(\infty^+) + \frac{B_w}{2\pi i} + \Pi - U(D)\right)_m k_m$$

$$+2\pi i \sum_{m=1}^{g} \frac{m}{2}k_m + \pi i \sum_{m=1}^{g} \left(-\sum_{j=1}^{g} b_{mj}k_m + \sum_{l=1}^{g} b_{ml}k_l k_m\right)\}$$

$$= \sum_{k\in Z^g} exp\{2\pi i \sum_{m=1}^{g} \left(U(\infty^+) + \frac{B_w}{2\pi i} + \Pi - U(D)\right)_m k_m$$

$$+\pi i \sum_{m=1}^{g} mk_m + \pi i \sum_{l,m=1}^{g} k_m b_{ml}(k_l - 1)\}.$$

Moreover, $Re(ib_{mm}) = -\infty$. Hence the k-th term in the sum is non-zero in the limit iff $k \in \{0,1\}^g$. Now

$$\theta(U(\infty^+) + d + \frac{B}{2\pi i} + \Pi) =$$

$$\sum_{k\in(0,1)^g} exp\{\left(\sum_{m=1}^{g} -log\beta(\infty^+, \lambda_m) + B_{wm} + 2\pi i\Pi_m + \sum_{j=1}^{g} log\beta(\mu_j, \lambda_m) + \pi i \sum_{m=1}^{g} m\right) k_m$$

$$- \sum_{l,m=1}^{g} k_m(k_l - 1)log\beta_{ml}\}$$

$$= \sum_{k\in(0,1)^g} \frac{\prod_{s,j=1}^{g} \beta(\lambda_s, \mu_j)^{k_s} \prod_{s,j=1}^{g} \beta_{sj}^{k_s}}{\prod_{s=1}^{g} \beta(\lambda_s, \lambda)^{k_s} \prod_{s,j=1}^{g} \beta_{sj}^{k_s k_j}} e^{\pi i \sum_{s=1}^{g} sk_s} e^{2\pi i \sum_{s=1}^{g} \Pi_s k_s} \prod_{s=1}^{g} C_s^{k_s}.$$

where $C_m = e^{B_{wm}}$

A similar computation yields the following results

$$\theta(U(\infty^+) + d + \frac{B}{2\pi i} + \Pi) =$$

$$\sum_{k\in(0,1)^g} \frac{\prod_{s,j=1}^g \beta(\lambda_s,\mu_j)^{k_s} \prod_{s,j=1}^g \beta_{sj}^{k_s}}{\prod_{s=1}^g \beta(\lambda_s,\infty^+)^{k_s} \prod_{s,j=1}^g \beta_{sj}^{k_s k_j}} e^{\pi i \sum_{s=1}^g s k_s} e^{2\pi i \sum_{s=1}^g \Pi_s k_s} \prod_{s=1}^g C_s^{k_s}, \tag{6.2}$$

$$\theta(U(\infty^+)+d) =$$

$$\sum_{k\in(0,1)^g} \frac{\prod_{s,j=1}^g \beta(\lambda_s,\mu_j)^{k_s} \prod_{s,j=1}^g \beta_{sj}^{k_s}}{\prod_{s=1}^g \beta(\lambda_s,\infty^+)^{k_s} \prod_{s,j=1}^g \beta_{sj}^{k_s k_j}} e^{\pi i \sum_{s=1}^g s k_s}, \tag{6.3}$$

$$\theta(-U(\infty^+)+d+\frac{B}{2\pi i}+\Pi) =$$

$$\sum_{k\in(0,1)^g} \frac{\prod_{s=1}^g \beta(\lambda_s,\infty^+)^{k_s} \prod_{s,j=1}^g \beta(\lambda_s,\mu_j)^{k_s} \prod_{s,j=1}^g \beta_{sj}^{k_s}}{\prod_{s,j=1}^g \beta_{sj}^{k_s k_j}}$$

$$e^{\pi i \sum_{s=1}^g s k_s} e^{2\pi i \sum_{s=1}^g \Pi_s k_s} \prod_{s=1}^g C_s^{k_s}, \tag{6.4}$$

$$\theta(-U(\infty^+)+d) =$$

$$\sum_{k\in(0,1)^g} \frac{\prod_{s=1}^g \beta(\lambda_s,\infty^+)^{k_s} \prod_{s,j=1}^g \beta(\lambda_s,\mu_j)^{k_s} \prod_{s,j=1}^g \beta_{sj}^{k_s}}{\prod_{s,j=1}^g \beta_{sj}^{k_s k_j}} e^{\pi i \sum_{s=1}^g s k_s}. \tag{6.5}$$

Now let us consider what happens with the integral W under the limiting procedure. We have

$$W = \sum_{s=1}^g \alpha_s \left(W(p_s,\lambda) - W(p_s^*,\lambda)\right), \tag{6.6}$$

$$W(p,\lambda) = \int_{i\bar{\xi}}^{\lambda} \frac{h(s)+h(p)+h'(p)(s-p)}{(s-p)^2 h(s)} ds,$$

$$W(p^*,\lambda) = \int_{i\bar{\xi}}^{\lambda} \frac{h(s)-h(p)-h'(p)(s-p)}{(s-p)^2 h(s)} ds,$$

$$h(s) = \sqrt{(s+i\xi)(s-i\bar{\xi})\prod_{i=1}^g (s-a_i)(s-b_i)},$$

where p_s and α_s are real numbers. Under the limiting procedure,

$$\alpha(W(p,\lambda)-W(p^*,\lambda)) = 2\alpha \frac{h(\lambda)}{h(p)(p-\lambda)},$$

so,

$$W = 2\sum_{s=1}^{g} \alpha_s \frac{h(\lambda)}{h(p_s)(p_s - \lambda)}$$

and

$$(B_W)_k = 4\sum_{s=1}^{g} \alpha_s \frac{h(b_k)}{h(p_s)(p_s - b_k)}.$$

We choose the behavior of parameters p_s and α_s according to

$$p_s \to \lambda_s$$

and

$$\alpha_s = \omega_s(p_s - b_s), \tag{6.7}$$

for some real numbers ω_s, as yet unspecified. Then $W(p,\lambda) \to 0$ except for finite number of points $\lambda = \lambda_1, ..., \lambda_g$ and $(B_W)_k \to 4\omega_k$.

Note that the function Φ has a singular limit under the limiting procedure. However, we are only interested in the value at infinity.

According to the above considerations we have a class of solutions of EVE of the following form: Let

$$C_k \equiv e^{4\omega_k}, \tag{6.8}$$

$$\mathcal{P}_1 \equiv \prod_{s,j=1}^{g} \beta_{sj}^{k_s},$$

$$\mathcal{P}_2 \equiv \prod_{s,j=1}^{g} \beta_{sj}^{k_s k_j},$$

$$\mathcal{P}_3 \equiv \prod_{s,j=1}^{g} \beta(\lambda_s, \mu_j)^{k_s},$$

$$\mathcal{P}_4 \equiv \prod_{s=1}^{g} \beta(\lambda_s, \infty^+)^{k_s},$$

$$\mathcal{P}_5 \equiv \prod_{s=1}^{g} C_s^{k_s},$$

$$\mathcal{S}_1 \equiv \sum_{s=1}^{g} s k_s,$$

$$\mathcal{S}_2 \equiv \sum_{s=1}^{g} \Pi_s k_s,$$

then the Ernst potential obtained after taking the limits

$$a_i - b_i \to 0, \;\; a_i \to \lambda_i, \;\; i = 1, ..., g,$$

$$p_s \to \lambda_s, \;\; \alpha_s = \omega_s(p_s - b_s),$$

is given by

$$\overline{\mathcal{E}} = \pm \frac{\sum\limits_{k\in\{0,1\}^g} \frac{\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5}{\mathcal{P}_2\mathcal{P}_4} e^{\pi i(\mathcal{S}_1+2\mathcal{S}_2)}}{\sum\limits_{k\in\{0,1\}^g} \frac{\mathcal{P}_1\mathcal{P}_3}{\mathcal{P}_2\mathcal{P}_4} e^{\pi i\mathcal{S}_1}} \frac{\sum\limits_{k\in\{0,1\}^g} \frac{\mathcal{P}_1\mathcal{P}_3\mathcal{P}_4}{\mathcal{P}_2} e^{\pi i\mathcal{S}_1}}{\sum\limits_{k\in\{0,1\}^g} \frac{\mathcal{P}_1\mathcal{P}_3\mathcal{P}_4\mathcal{P}_5}{\mathcal{P}_2} e^{\pi i(\mathcal{S}_1+2\mathcal{S}_2)}}, \tag{6.9}$$

where $\mathcal{P}_i$, $\mathcal{S}_i$ depend on k, $\Pi_s = -\frac{1}{4}$ and the parameters λ_i, C_k, μ_j are at our disposal.

Although the RH Problem itself becomes singular in the limit, $\mathcal{E}$, which is obtained from behavior of Ψ at infinity, remains finite, and continues to solve the Ernst equation, and hence produces a solution of EVE.

CHAPTER 7

Force.

The general relativistic meaning of “force ”is not clear , and the purpose of this chapter is to explain what is meant by “force ”and to give a recipe for the computation of the force in a given space-time. An essential part of the argument involves an analysis of the boundary conditions on the axis of symmetry.

By definition [MiWe] of the stress energy tensor, in a region filled with fluid, $T_{xy}dS^y$ (no summation) is the x component of the force acting through a small area dS^y with normal pointing in the y direction. The x component of the force acting on a three dimensional domain is given by the flux of the vector (T_{xx}, T_{xy}, T_{xz}) through the domain's boundary. We use the same concept for the definition of force in the vacuum case. For two bodies, $T_{ij} = 0$ except along the line between them, where it is singular. To compute the force which one of the bodies exerts on the other we must take the singularity in the nature of the force into account. We surround the body by a two dimensional surface and show how to compute the force by integrating over a small part of the surface near the axis of symmetry where T_{ij} is singular.

For the geometry generated by the Ernst potential of Chapter 6, space-time has a so-called conical singularity on the axis of ϕ-symmetry between the bodies (cf. [Wein],[BachWeyl]). We follow the computation in [Wein]. Suppose C_ρ is the coordinate circle $\{\rho = const\}$ in the ρ, ϕ plane centered at $\rho = 0$. For a smooth space ([ONeil], p. 359-360) the ratio of the circumference to the radius of such a circle goes to 2π as ρ goes to zero. The conical singularity reflects the fact that the limit is not equal to 2π on the part of the axis between the bodies.

Suppose, the ρ, ϕ part of the metric is given by

$$ds^2 = \rho^2 e^u d\phi^2 + e^{2\lambda} d\rho^2,$$

for some functions $u = u(\rho, z)$, $\lambda = \lambda(\rho, z)$. Then

$$\text{Circumference of } C_\rho = \int_{\phi=0}^{\phi=2\pi} \sqrt{ds^2} = \int_{\phi=0}^{\phi=2\pi} \sqrt{\rho^2 e^u} d\phi =$$

$$= \rho e^{\frac{u(\rho)}{2}} 2\pi,$$

and

$$\text{Radius of } C_\rho = \int_{\rho=0}^{\rho} e^\lambda d\rho = \rho e^\lambda|_0^\rho - \int_0^\rho \rho d e^\lambda.$$

Let β be defined through the relation $\lambda = \frac{1}{2}(u + \beta)$. Then the limit of Circumference/Radius when $\rho \to 0$ is given by

$$\frac{2\pi}{e^{\frac{1}{2}\beta(\rho=0)}},$$

which measures the angular deficiency.

The axis is clearly divided into three pieces by the positions of the two bodies. For the geometry at hand, we find that β is piecewise constant (away from the bodies, see below). Furthemore, it contains a constant of integration which insures that we may take $\beta = 0$ on unbounded components of the axis. However, in general, β is not zero on the bounded component.

We now compute the expression for the "force ". We integrate the zz component of the stress energy tensor over the coordinate disk $\{0 \leq \rho \leq \rho_0, 0 \leq$

$\phi \leq 2\pi\}$ in the ρ, ϕ plane centered at the axis of the symmetry and let the radius ρ_0 of the disk go to zero. The limiting value of the integral is by definition the force. By the Einstein field equations, the zz component of the stress energy tensor is given by

$$T_{zz} = \frac{1}{8\pi}\left(R_{zz} - \frac{1}{2}Rg_{zz}\right). \tag{7.1}$$

The quantities R_{zz} and R in the RHS of (7.1) denote the zz component of the Ricci tensor and the scalar curvature respectively, and are sums of terms of the form $R^i{}_{jij}$. As we see below $R^{\phi}{}_{\rho\phi\rho}$ and $R^{\rho}{}_{\phi\rho\phi}$ have a Dirac δ-function like singularity on the axis. This reflects the presence of the conical singularity in the $\{\rho, \phi\}$ plane discussed above. Since there is no singularity of the metric in the other coordinate planes the remaining $R^i{}_{jij}$ terms are regular.

We use the notation

$$t = x^0, \phi = x^1, \rho = x^2, z = x^3.$$

For the above geometry, the terms

$$R_{00} = R^1{}_{010} + R^2{}_{020} + ...,$$

$$R_{33} = R^0{}_{303} + ...$$

are regular. However,

$$R_{11} = R^2{}_{121} + regular,$$

$$R_{22} = R^1{}_{212} + regular,$$

$$R = R^i_i = R^{12}{}_{12} + R^{21}{}_{21} + regular = 2R^{12}{}_{12} + regular,$$

and so

$$R = 2Kg^{22} + regular,$$

where $K = R^1{}_{212}$ is the Gaussian curvature. For our metric, $g_{33} = 1/g^{22}$ (see (1.12)). Therefore,

$$T_{zz} = T_{33} = \frac{1}{8\pi}\left(R_{33} - \frac{1}{2}Rg_{33}\right) =$$

$$= -\frac{1}{16\pi}g_{33}R + regular = -\frac{1}{8\pi}K + regular.$$

We need the Gauss-Bonnet formula:

$$\int_{B_\rho} KdA = 2\pi - \int_{C_\rho} k_g ds.$$

where B_ρ is a coordinate disk in the ρ, ϕ plane, C_ρ is a coordinate circle and k_g is the geodesic curvature of C_ρ. A general procedure for the computation of the geodesic curvature of a curve is as follows. Suppose the metric is given by the formula

$$ds^2 = Edw^2 + Gdv^2,$$

where w and v are local parameters on the manifold. Also, suppose that a curve is given parametrically by the equations

$$w = w(t), v = v(t),$$

where t is the parameter. Then the geodesic curvature is given by the relation

$$k_g = \left(Ew_t^2 + Gv_t^2\right)^{-\frac{1}{2}}\left(f_t + \frac{1}{2\sqrt{EG}}\left(G_w v_t - E_v w_t\right)\right),$$

where the function f is given in turn by

$$f = arctg\left(\sqrt{\frac{G}{E}}\frac{dv}{dw}\right).$$

For our particular case we identify parameters v and w as

$$v = \rho, w = \phi.$$

The circle C_ρ has the parametrization

$$C_\rho = \{v = \rho = const, w = -t \text{ for } t \in [0, 2\pi]\}.$$

The metric is given by

$$ds^2 = e^{2\lambda} d\rho^2 + \rho^2 e^u d\phi^2,$$

which allows us to identify E and G as

$$E = \rho^2 e^u,$$

$$G = e^{2\lambda}.$$

Therefore we compute the geodesic curvature k_g of the curve C_ρ as follows:

$$w_t = -1, v_t = 0,$$

$$k_g = \left(Ew_t^2 + Gv_t^2\right)^{-\frac{1}{2}}\left(\phi_t + \frac{1}{2\sqrt{EG}}\left(G_w v_t - E_v w_t\right)\right) =$$

$$= \left(\rho^2 e^u\right)^{\frac{1}{2}}\left(\frac{1}{2\sqrt{\rho^2 e^u e^{2\lambda}}}\left(\frac{d}{d\rho}\left(\rho^2 e^u\right)\right)\right)$$

$$= \frac{1}{\rho e^{u/2}} \left(\frac{1}{2} \frac{1}{\rho e^{u/2} e^{\lambda}} \left(2\rho e^{u} + \rho^{2} e^{u} \frac{du}{d\rho} \right) \right) \quad \text{(Note: the } \phi\text{-derivative is zero)}$$

$$= \frac{1}{\rho^{2} e^{u} e^{\lambda}} \frac{1}{2} \left(2\rho e^{u} + \rho^{2} e^{u} \frac{du}{d\rho} \right) =$$

$$= e^{-\lambda} \left(\frac{1}{\rho} + \frac{1}{2} \frac{du}{d\rho} \right).$$

Hence,

$$\int_{C_{\rho}} k_{g} ds =$$

$$= \int_{0}^{2\pi} e^{-\lambda} \left(\frac{1}{\rho} + \frac{1}{2} \frac{du}{d\rho} \right) \sqrt{\rho^{2} e^{u} d\phi^{2}} =$$

$$= \int_{0}^{2\pi} e^{-\lambda} \left(\frac{1}{\rho} + \frac{1}{2} \frac{du}{d\rho} \right) \rho e^{\frac{u}{2}} d\phi =$$

$$= 2\pi e^{-\frac{\beta}{2}} \left(1 + \frac{1}{2} \rho \frac{du}{d\rho} \right).$$

Thus

$$F = Lim_{\rho \to 0} \int_{B_{\rho}} T_{zz} dA = -\frac{1}{8\pi} Lim_{\rho \to 0} \int_{B_{\rho}} K dA =$$

$$= -\frac{1}{8\pi} Lim_{\rho \to 0} \left(2\pi - \int_{C_{\rho}} k_{g} ds \right) = -\frac{1}{8\pi} Lim_{\rho \to 0} \left(2\pi - 2\pi e^{-\frac{\beta}{2}} \left(1 + \frac{1}{2} \rho \frac{du}{d\rho} \right) \right)$$

$$= -\frac{1}{4} \left(1 - e^{-\frac{\beta(0)}{2}} \right).$$

Comparing

$$ds^{2} = \rho^{2} e^{u} d\phi^{2} + e^{2\lambda} d\rho^{2}$$

with the notation of the formula (1.12),

$$ds^{2} = e^{-2U} (e^{2K} (d\rho^{2} + dz^{2}) + \rho^{2} d\phi^{2}) - e^{2U} (dt + A d\phi)^{2} =$$

$$= -e^{2U}dt^2 - 2Ae^{2U}dtd\phi + \left(e^{-2U}\rho^2 - e^{2U}A^2\right)d\phi^2 + e^{2K-2U}\left(d\rho^2 + dz^2\right),$$

and we see that

$$\rho^2 e^u = e^{-2U}\rho^2 - e^{2U}A^2,$$

$$e^{2\lambda} = e^{2K-2U},$$

$$\lambda = \frac{1}{2}(u + \beta).$$

Thus,

$$\beta = 2(K - U) - u,$$

where u is given by the expression

$$e^u = e^{-2U} - \frac{e^{2U}A^2}{\rho^2}.$$

Hence

$$e^{-\frac{\beta(0)}{2}} = Lim_{\rho\to 0}e^{-\frac{1}{2}(2(K-U)-u)} = e^{U-K+\frac{u}{2}} =$$

$$= Lim_{\rho\to 0}e^{U-K}\sqrt{e^{-2U} - \frac{e^{2U}A^2}{\rho^2}}$$

$$= Lim_{\rho\to 0}e^{-K}\sqrt{1 - \frac{e^{4U}A^2}{\rho^2}}.$$

We see that if A does not vanish sufficiently fast then the expression for the force becomes singular. The angle difficiency discussed earlier also becames singular.

For a physically acceptable geometry we require that

$$\frac{e^{2U}A}{\rho} < 1$$

as $\rho \to 0$. According to the expressions,

$$\partial_z A = -\frac{\rho}{v^2}\partial_\rho \Phi, \tag{7.2}$$

$$\partial_\rho A = \frac{\rho}{v^2}\partial_z \Phi, \tag{7.3}$$

we have $\partial_z A = O(\rho^3)$ and $\partial_\rho A = O(\rho)$ provided that $\frac{\partial \Phi}{v^2}$ is not singular on the axis. We will see later that the solution for the geometry at hand does have this property. So, $A = A_0 + O(\rho^2)$ where $A_0 = A_0(z)$ is the limiting value of A as $\rho \to 0$.

In Chapter 9 we will show that $A_0 = A_0(z) = 0$ for all z on the axis away from the event horizons surrounding each of the bodies. Thus, the term $\frac{e^{4U}A^2}{\rho^2}$ drops out and we arrive at the following formula for the "force ":

$$Force = \frac{1}{4}\left(Lim_{\rho\to 0}e^{-K} - 1\right). \tag{7.4}$$

Here the limit $\rho \to 0$ is taken for any z on the axis between the two bodies and away from the event horizons.

CHAPTER 8

Mass and Angular momentum.

In this chapter we discuss a method [Wald] to compute the precise values for the total mass and angular momentum of an isolated system in a stationary situation. This allows us to attach a precise physical meaning to the parameters of the solution coming from the RHP.

Suppose V is a Killing vector field. It is a classical result [Wald] that there is a conserved quantity associated with it given up to some multiplicative constant by the Komar integral

$$I(V) = \int_S *dV.$$

Here S is a boundary of the region of interest, V is a 1-form associated with the vector field V, dV is the external derivative of the 1-form and $*$ denotes the Hodge star operator.

In our problem we surround one of the bodies under consideration by a surface of rotation about the z axis. Let C be a contour with end points on the axis of symmetry and let S be the surface of rotation generated by C. Let K be the Killing vector field ∂_ϕ. Using the fact that $*dV_{ab} = \epsilon_{abcd}\nabla^c V^d$ we have

$$I(V) = 2\pi \int_C \epsilon_{abcd} K^b \nabla^c V^d dx^a.$$

We are interested in the computation of $I(K)$ and $I(T)$ for the Killing vectors $K = \partial_\phi$, $T = \partial_t$. We use metrics given by expressions (1.4)-(1.12) to compute the quantities $I(T)$ and $I(K)$. The results of the computation are as follows:

$$I(T) = 2\pi \int_C \left(-\rho \frac{\partial_z v}{v} + A\partial_\rho \Phi\right) d\rho + \left(\rho \frac{\partial_\rho v}{v} + A\partial_z \Phi\right) dz, \tag{8.1}$$

$$I(K) = 2\pi \int_C \left(-2A\frac{\rho}{v}\partial_z v + \left(\frac{\rho^2}{v^2} + A^2\right) \partial_\rho \Phi\right) d\rho +$$

$$\left(2A\left(-1 + \frac{\rho}{v}\partial_\rho v\right) + \left(\frac{\rho^2}{v} + A^2\right) \partial_z \Phi\right) dz. \tag{8.2}$$

We want to show that the quantities $I(K)$ and $I(T)$ are independent of the position of the contour C. Suppose first that the end points of C are fixed. The contour C may connect its end points in many different ways. We do not allow self intersections and require that C is smooth and away from the event horizon. A direct computation shows that the integrands of $I(K)$ and $I(T)$ are full differentials provided that the Ernst equation is satisfied.

We show next that $I(K)$ and $I(T)$ are independent of the position of the end points provided of course that the end points stay on the z-axis and away from event horizons. This property may be seen from (8.1)-(8.2). For example, since

$$I(K) = 2\pi \int_C dU$$

for some U, the expression in front of dz in (8.1) is a partial derivative of U with respect to z:

$$U_z = 2A\left(-1 + \frac{\rho}{v}\partial_\rho v\right) + \left(\frac{\rho^2}{v} + A^2\right) \partial_z \Phi.$$

Thus, if A vanishes on the z-axis then U_z vanishes on the z-axis. But the integral is equal to the difference of values of U at end points of C on the z-axis. So, the position of the contour C does not affect the result of integration provided that we stay away from the bodies.

We are considering two bodies located a large distance apart. For each of the bodies we choose a surface of revoution as above and compute $I(K)$ and $I(T)$. Naturally, if we fix the coordinates so that one body is at $z = 0$ and the other is at $z = R$, then near the first body we expect that

$$\mathcal{E} = \mathcal{E}_{Kerr}(m, a) + o(1), \tag{8.3}$$

as $R \to \infty$. Here $\mathcal{E}_{Kerr}(m, a)$ is the classical Kerr solution with the parameters a and m. We will see directly that our solution has this property.

For the Kerr solution we find

$$m = \frac{1}{8\pi} I(T)$$

and

$$ma = \frac{1}{16\pi} I(K).$$

For the quantities $I(K)$ and $I(T)$ computed for our solution, we define:

$$M = \frac{1}{8\pi} I(T)$$

and

$$MA = \frac{1}{16\pi} I(K).$$

Clearly, M and MA have the interpretation of mass and angular mometum of the body under consideration. By (8.3) it follows that

$$M = m + o(1),$$

$$A = a + o(1).$$

CHAPTER 9

Computations.

In this chapter we explain how to extract the Ernst potential for two rotating black holes using the degeneration procedure of Chapter 6. We also compute the space time function K (see 1.8,1.9,1.12) and the force (7.4). Formula (9.18) below is the main result of this thesis, indicating the effect of rotation on the higher order terms in the force between two gravitational bodies.

In chapter 6 we have shown that a class of solutions of EVE is given by the Ernst potential in (6.8)-(6.9).

Let us consider the computation using (6.8)-(6.9) with genus $g = 2$, which corresponds to a single body located at $\rho = z = 0$.

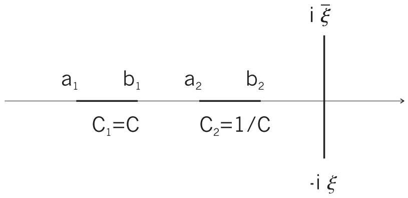

Figure 9.1: Kerr solution RHP.

Figure 9.1 shows the upper sheet of the surface $\mathcal{L}_0$ (see remarks after formula 4.9). Because $\xi = \rho + iz$, the moving cut intersects the real line at the point $\lambda = (z, 0)$. The non-moving cut $[a_1, b_1]$ collapses to a point $-\sigma$ and the non-moving cut $[a_2, b_2]$ collapses to a point σ. In the $\{\rho, z\}$-plane the event horizon of the body is the degenerate ellipse $[-\sigma, \sigma]$ on the z-line. It is essential that the moving cut is located to the right of the non-moving cuts because it affects the way we compute the $\mathcal{B}$-periods (see discussion around Figure 6.3).

If we choose the limit of the divisor D to be of the form $D = \{\infty^+, \infty^-\}$ then the Ernst potential computed according to (6.8)-(6.9) has the from

$$\overline{\mathcal{E}} = \frac{\beta(-\sigma,\infty)\,(i\,C_2\,\beta(-\sigma,\sigma) - \beta(\sigma,\infty)) - C_1\,(C_2 + i\,\beta(-\sigma,\sigma)\,\beta(\sigma,\infty))}{1 - i\,C_2\,\beta(-\sigma,\sigma)\,\beta(\sigma,\infty) + C_1\,\beta(-\sigma,\infty)\,(i\,\beta(-\sigma,\sigma) + C_2\,\beta(\sigma,\infty))}, \quad (9.1)$$

where the function β is given by (6.1). In the prolate spheroidal coordinates introduced in Chapter 5, the expressions for β are as follows:

$$\beta(\sigma,\infty) = -\frac{(1+X)\,(-1+Y)}{\sqrt{-1+X^2}\,\sqrt{-1+Y^2}},$$

$$\beta(-\sigma,\infty) = \frac{-1+X+Y-X\,Y}{\sqrt{-1+X^2}\,\sqrt{-1+Y^2}},$$

$$\beta(-\sigma,\sigma) = \frac{\sqrt{-1+X^2}}{\sqrt{-1+Y^2}}.$$

If we use the values $C_1 = C$, $C_2 = 1/C$ then we see that the Ernst potential derived by degenerating the $g = 2$ RHP becomes exactly the Ernst potential for the Kerr solution (5.4) introduced in Chapter 5 with the parameter p given by

$$p = \frac{1-C^2}{1+C^2}. \quad (9.2)$$

If we place the moving cut to the left of the non-moving cuts then we obtain the Kerr solution rotating in the opposite direction. This is expected because the RHP corresponding to Figure 9.2 may be obtained from the RHP corresponding to Figure 9.1 by the substitution $z \to -z$.

According to (5.3) and (5.4) the Schwartzchild solution may be obtained if we let $C \to 0$ in the procedure for the extraction of the Kerr solution.

The above two degeneration results are also discussed in [Korot].

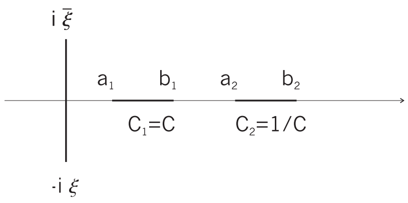

Figure 9.2: Kerr solution with the rotation in different direction.

A similar computation may be done for $g = 4$. We obtain the Weyl solution if we consider the case $g = 4$ and degenerate to the case $D = \{\infty^+, \infty^-, \infty^+, \infty^-\}$, $C_1 = A$, $C_2 = 1/A$, $C_3 = A$, $C_4 = 1/A$, $\lambda_1 = -m$, $\lambda_2 = m$, $\lambda_3 = -m + R$, $\lambda_4 = m + R$ and then let $A \to 0$.

As noted above, the Schwartzshild solution is a limiting case of the Kerr solution as $C \to 0$, but also the Weyl solution may be obtained from the Schwartzchild solution by adding two non-moving cuts indentical to those already present. This observation motivates us to look for a solution corresponding to two rotating black holes by adding two more cuts to the RHP for the Kerr solution or, equivalently, by letting A be arbitrary in the Weyl RHP. It is important to place the moving cut between the second and the third non-moving cuts (see Figure 9.3): this way we obtain the Kerr potential which corresponds to black holes rotating in opposite directions. After degeneration we obtain the Ernst potential

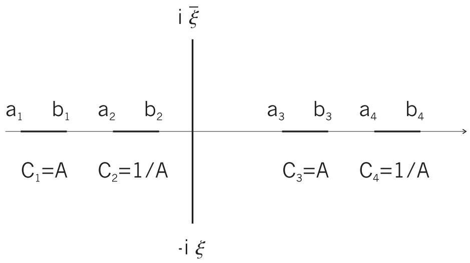

Figure 9.3: Two rotating black holes RHP.

for two rotating black holes from (6.8)- (6.9) by chosing the parameters $g = 4$,

$D = \{\infty^+, \infty^-, \infty^+, \infty^-\}$, $C_1 = A$, $C_2 = 1/A$, $C_3 = A$, $C_4 = 1/A$, $\lambda_1 = -\sigma$, $\lambda_2 = \sigma$, $\lambda_3 = -\sigma + R$ and $\lambda_4 = \sigma + R$. The computation is conceptually straightforward. However, in this case it becames so complex that one cannot proceed without using some software for symbolic manipulation.

Let us introduce a function γ by the relation

$$\gamma(\lambda_i, \lambda_j) = -i\,\rho\,\beta(\lambda_i, \lambda_j),$$

so, according to (6.1), $\gamma(\lambda_i, \lambda_j, \rho^2, z)$ is real polynomial function of ρ^2, z, $\sqrt{(z-\lambda_i)^2+\rho^2}$ and $\sqrt{(z-\lambda_j)^2+\rho^2}$ for real λ_i, λ_j. In terms of γ the solution v and Φ may be expressed as

$$v = \frac{v_n}{v_d}, \tag{9.3}$$

$$\begin{aligned}
v_n = {} & \Big(A^2\,\rho^6 + F_{13} - A^2\,F_9\,F_{14} + A^2\,\gamma(\lambda_1,\infty)\,\left(F_{11}\,F_{15} + F_{10}\,\gamma(\lambda_2,\infty)\right)\Big) \\
& \Big(A^2\,\left(\rho^6 + A^2\,F_{13} - F_9\,F_{14}\right) + \gamma(\lambda_1,\infty)\,\left(F_{12}\,F_{15} + A^2\,F_{10}\,\gamma(\lambda_2,\infty)\right)\Big) + \\
& A^2\,\Big(A^2\,F_1\,F_{16} + \gamma(\lambda_1,\infty)\,\left(F_4\,F_{17} + F_3\,\gamma(\lambda_1,\lambda_2)\right) + \rho^2\,F_2\,\gamma(\lambda_4,\lambda_3)\Big) \\
& \Big(F_5\,F_{16} + \gamma(\lambda_1,\infty)\,\left(F_8\,F_{17} + A^2\,F_7\,\gamma(\lambda_1,\lambda_2)\right) + \rho^2\,F_6\,\gamma(\lambda_4,\lambda_3)\Big),
\end{aligned}$$

$$\begin{aligned}
v_d = {} & \Big(A^2\,\rho^6 + F_{13} - A^2\,F_9\,F_{14} + A^2\,\gamma(\lambda_1,\infty)\,\left(F_{11}\,F_{15} + F_{10}\,\gamma(\lambda_2,\infty)\right)\Big)^2 + \\
& A^2\,\Big(F_5\,F_{16} + \gamma(\lambda_1,\infty)\,\left(F_8\,F_{17} + A^2\,F_7\,\gamma(\lambda_1,\lambda_2)\right) + \rho^2\,F_6\,\gamma(\lambda_4,\lambda_3)\Big)^2,
\end{aligned}$$

$$\Phi = \frac{A\Phi_n}{\Phi_d}, \tag{9.4}$$

$$\begin{aligned}
\Phi_n = {} & \Big(A^2\,\rho^6 + F_{13} - A^2\,F_9\,F_{14} + A^2\,\gamma(\lambda_1,\infty)\,\left(F_{11}\,F_{15} + F_{10}\,\gamma(\lambda_2,\infty)\right)\Big) \\
& \Big(A^2\,F_1\,F_{16} + \gamma(\lambda_1,\infty)\,\left(F_4\,F_{17} + F_3\,\gamma(\lambda_1,\lambda_2)\right) + \rho^2\,F_2\,\gamma(\lambda_4,\lambda_3)\Big) -
\end{aligned}$$

$$\left(A^2\,\left(\rho^6 + A^2\,F_{13} - F_9\,F_{14}\right) + \gamma(\lambda_1,\infty)\,\left(F_{12}\,F_{15} + A^2\,F_{10}\,\gamma(\lambda_2,\infty)\right)\right)$$

$$\left(F_5\,F_{16} + \gamma(\lambda_1,\infty)\,\left(F_8\,F_{17} + A^2\,F_7\,\gamma(\lambda_1,\lambda_2)\right) + \rho^2\,F_6\,\gamma(\lambda_4,\lambda_3)\right),$$

$$\Phi_d = \left(A^2\,\rho^6 + F_{13} - A^2\,F_9\,F_{14} + A^2\,\gamma(\lambda_1,\infty)\,\left(F_{11}\,F_{15} + F_{10}\,\gamma(\lambda_2,\infty)\right)\right)^2 +$$

$$A^2\left(F_5\,F_{16} + \gamma(\lambda_1,\infty)\,\left(F_8\,F_{17} + A^2\,F_7\,\gamma(\lambda_1,\lambda_2)\right) + \rho^2\,F_6\,\gamma(\lambda_4,\lambda_3)\right)^2,$$

$$F_1 = -\left(\gamma(\lambda_3,\infty)\,\gamma(\lambda_3,\lambda_1)\,\gamma(\lambda_4,\infty)\,\gamma(\lambda_4,\lambda_1)\right) + \rho^2\,\gamma(\lambda_3,\lambda_2)\,\gamma(\lambda_4,\lambda_2),$$

$$F_2 = -\left(\gamma(\lambda_3,\infty)\,\gamma(\lambda_3,\lambda_1)\,\gamma(\lambda_3,\lambda_2)\right) + A^2\,\gamma(\lambda_4,\infty)\,\gamma(\lambda_4,\lambda_1)\,\gamma(\lambda_4,\lambda_2),$$

$$F_3 = -\left(\rho^2\,\gamma(\lambda_3,\lambda_1)\,\gamma(\lambda_4,\lambda_1)\right) + \gamma(\lambda_3,\infty)\,\gamma(\lambda_3,\lambda_2)\,\gamma(\lambda_4,\infty)\,\gamma(\lambda_4,\lambda_2),$$

$$F_4 = -\left(A^2\,\gamma(\lambda_3,\lambda_1)\,\gamma(\lambda_3,\lambda_2)\,\gamma(\lambda_4,\infty)\right) + \gamma(\lambda_3,\infty)\,\gamma(\lambda_4,\lambda_1)\,\gamma(\lambda_4,\lambda_2),$$

$$F_5 = \gamma(\lambda_3,\infty)\,\gamma(\lambda_3,\lambda_1)\,\gamma(\lambda_4,\infty)\,\gamma(\lambda_4,\lambda_1) - \rho^2\,\gamma(\lambda_3,\lambda_2)\,\gamma(\lambda_4,\lambda_2),$$

$$F_6 = A^2\,\gamma(\lambda_3,\infty)\,\gamma(\lambda_3,\lambda_1)\,\gamma(\lambda_3,\lambda_2) - \gamma(\lambda_4,\infty)\,\gamma(\lambda_4,\lambda_1)\,\gamma(\lambda_4,\lambda_2),$$

$$F_7 = \rho^2\,\gamma(\lambda_3,\lambda_1)\,\gamma(\lambda_4,\lambda_1) - \gamma(\lambda_3,\infty)\,\gamma(\lambda_3,\lambda_2)\,\gamma(\lambda_4,\infty)\,\gamma(\lambda_4,\lambda_2),$$

$$F_8 = \gamma(\lambda_3,\lambda_1)\,\gamma(\lambda_3,\lambda_2)\,\gamma(\lambda_4,\infty) - A^2\,\gamma(\lambda_3,\infty)\,\gamma(\lambda_4,\lambda_1)\,\gamma(\lambda_4,\lambda_2),$$

$$F_9 = \gamma(\lambda_3,\lambda_2)\,\gamma(\lambda_4,\infty)\,\gamma(\lambda_4,\lambda_1) + \gamma(\lambda_1,\lambda_2)\,\gamma(\lambda_2,\infty)\,\gamma(\lambda_4,\lambda_3),$$

$$F_{10} = \rho^2\,\gamma(\lambda_3,\infty)\,\gamma(\lambda_4,\infty) - \gamma(\lambda_3,\lambda_1)\,\gamma(\lambda_3,\lambda_2)\,\gamma(\lambda_4,\lambda_1)\,\gamma(\lambda_4,\lambda_2),$$

$$F_{11} = A^2\,\gamma(\lambda_3,\infty)\,\gamma(\lambda_3,\lambda_2)\,\gamma(\lambda_4,\lambda_1) - \gamma(\lambda_3,\lambda_1)\,\gamma(\lambda_4,\infty)\,\gamma(\lambda_4,\lambda_2),$$

$$F_{12} = \gamma(\lambda_3,\infty)\,\gamma(\lambda_3,\lambda_2)\,\gamma(\lambda_4,\lambda_1) - A^2\,\gamma(\lambda_3,\lambda_1)\,\gamma(\lambda_4,\infty)\,\gamma(\lambda_4,\lambda_2),$$

$$F_{13} = \gamma(\lambda_1,\lambda_2)\,\gamma(\lambda_2,\infty)\,\gamma(\lambda_3,\lambda_2)\,\gamma(\lambda_4,\infty)\,\gamma(\lambda_4,\lambda_1)\,\gamma(\lambda_4,\lambda_3),$$

$$F_{14} = \gamma(\lambda_3, \infty)\,\gamma(\lambda_3, \lambda_1)\,\gamma(\lambda_4, \lambda_2),$$

$$F_{15} = \gamma(\lambda_1, \lambda_2)\,\gamma(\lambda_4, \lambda_3),$$

$$F_{16} = \gamma(\lambda_1, \lambda_2)\,\gamma(\lambda_2, \infty),$$

$$F_{17} = \gamma(\lambda_2, \infty)\,\gamma(\lambda_4, \lambda_3).$$

The solution has the following

Properties:

Property 1. Smoothness.

The solution functions v and Φ are real analytic functions of $\{\rho, z\}$ variables away from the branch points $\{\lambda_i\}$.

Indeed, one can see from the above expressions that the functions v and Φ are of the form

$$F(\lambda_1, ..., \lambda_4, \sqrt{(z-\lambda_1)^2+\rho^2}, ..., \sqrt{(z-\lambda_4)^2+\rho^2}, \rho^2, z) \tag{9.5}$$

where $\{\lambda_i\}_{i=1,...,4}$ are positions of the end points of the event horizons on the z-line and F is a rational function of its arguments which is, of course, different for v and Φ.

Property 2. Local closeness to Kerr solution.

Suppose $\lambda_1 = -\sigma$, $\lambda_2 = \sigma$, $\lambda_3 = -\sigma + R$ and $\lambda_4 = \sigma + R$. The decompositon of v and Φ in powers of $\frac{1}{R}$ as $R \to +\infty$ and ρ and z in a neighboorhood of either one of the bodies is of the form

$$v = v_{Kerr} + v_1\frac{1}{R} + O(\frac{1}{R^2}),$$

$$\Phi = \Phi_{Kerr} + \Phi_1 \frac{1}{R} + O(\frac{1}{R^3}),$$

where v_{Kerr} and Φ_{Kerr} are the real and imaginary parts of the Ernst potential for the Kerr solution (see formula (5.4)). The above expansion for v is valid in any compact set of $\{\rho, z\}$ away from the z-axis because the expansions of v_n and v_d are of the form

$$v_n = \frac{64\,p^2\,(-1+X^2)^4\,(-1+Y)^6\,(-1+Y^2+p^2\,(X^2-Y^2))\,\sigma^6}{q^4} R^6$$

$$-(8\left(-1+A^2\right)^2(-1+X)\,X\,(1+X)\left(-1+X^2\right)^3(-1+Y)^6\,(-2+3\,Y)$$

$$\left(-1+X^2+A^4\left(-1+X^2\right)-2\,A^2\left(1+X^2-2\,Y^2\right)\right)\sigma^7)R^5 + O(R^4),$$

$$v_d = \frac{64\,p^2\,(-1+X^2)^4\,(-1+Y)^6\,(1+2\,p\,X+Y^2+p^2\,(X^2-Y^2))\,\sigma^6}{q^4} R^6 + O(R^5),$$

so the highest terms of the expansions are not vanishing away from the z-axis.

Now

$$v_{Kerr} = \frac{-1+X^2+A^4\,(-1+X^2)-2\,A^2\,(1+X^2-2\,Y^2)}{A^4\,(-1+X)^2+(1+X)^2+A^2\,(2-2\,X^2+4\,Y^2)},$$

$$v_1 = 2\sigma(1+A^2)(A^8\,(-1+X)^3\,(1+X)+(-1+X)\,(1+X)^3$$

$$-4\,A^2\,(1+X)\left(1+X^3-2\,Y-2\,X\,(-1+Y)\,Y\right)$$

$$-4\,A^6\,(-1+X)\left(-1+X^3+2\,Y-2\,X\,(-1+Y)\,Y\right)$$

$$+A^4\left(-6+6\,X^4+16\,Y-16\,X^2\,(-1+Y)\,Y-32\,Y^3+16\,Y^4\right))/$$

$$(\left(-1+A^2\right)\left(A^4\,(-1+X)^2+(1+X)^2+A^2\left(2-2\,X^2+4\,Y^2\right)\right)^2),$$

$$\Phi_{Kerr} = \frac{-4\,A\,(1+A^2)\,Y}{A^4\,(-1+X)^2 + (1+X)^2 + A^2\,(2-2\,X^2+4\,Y^2)},$$

$$\Phi_1 = 8\,A\,\left(1+A^2\right)\,\sigma(\left(-1+A^2\right)^3 X^3 + \left(-1+A^2\right)^2\,\left(1+A^2\right)\,X^2\,(-2+Y)$$

$$-\left(-1+A^2\right)\,X\,\left(-1+2\,Y+A^4\,(-1+2\,Y)+A^2\,\left(-2+4\,Y-4\,Y^2\right)\right)+$$

$$\left(1+A^2\right)\,Y\,\left(1+A^4+2\,A^2\,\left(1-4\,Y+2\,Y^2\right)\right))$$

$$/(\left(-1+A^2\right)\,\left(A^4\,(-1+X)^2+(1+X)^2+A^2\,\left(2-2\,X^2+4\,Y^2\right)\right)^2).$$

Note that the numerator of v_1 is divisable by the numerator of v_{Kerr}. Hence, the function v_1 vanishes at the ergosphere of the Kerr solution.

One can verify by direct substitution that the functions v and Φ given by these expansions satisfy the Ernst equation up to order of $1/R$ as high as the computer permits. The author has verified it up to fourth order.

To obtain the above expansions for the solution one has to position the origin of the $\{\rho, z\}$ coordinate system so that the $\{\lambda_i\}_{i=1,\ldots,4}$ are $\{-\sigma, \sigma, R-\sigma, R+\sigma\}$. We switch to the coordinate system $\{X, Y\}$ and compute the expansions for γ functions in terms of $\frac{1}{R}$. Then we combine the expansions for γ's in agreement with (9.3) and (9.4).

The expansions for γ are as follows

$$\gamma(\lambda_1, \infty) = -\left((-1+X)\,(-1+Y)\,\sigma\right),$$

$$\gamma(\lambda_2, \infty) = -\left((1+X)\,(-1+Y)\,\sigma\right),$$

$$\gamma(\lambda_3, \infty) = 2\,R - 2\,(1+X\,Y)\,\sigma - \frac{(-1+X^2)\,(-1+Y^2)\,\sigma^2}{2\,R}$$

$$-\frac{(-1+X^2)\,(1+X\,Y)\,(-1+Y^2)\,\sigma^3}{2\,R^2} + O(\frac{1}{R})^3,$$

$$\gamma(\lambda_4,\infty) = 2\,R - 2\,(-1+X\,Y)\,\sigma - \frac{(-1+X^2)\,(-1+Y^2)\,\sigma^2}{2\,R}$$

$$-\frac{(-1+X^2)\,(-1+X\,Y)\,(-1+Y^2)\,\sigma^3}{2\,R^2} + O(\frac{1}{R})^3,$$

$$\gamma(\lambda_1,\lambda_2) = \left(-1+X^2\right)\,\sigma,$$

$$\gamma(\lambda_2,\lambda_1) = \sigma - X^2\,\sigma,$$

$$\gamma(\lambda_4,\lambda_2) = (1+X)\,(-1+Y)\,\sigma +$$

$$\frac{(1+X)\,(X-Y)\,(-1+Y)\,\sigma^2}{R} + \frac{(-1+X^2)\,(X-Y)\,(-1+Y^2)\,\sigma^3}{2\,R^2} + O(\frac{1}{R})^3,$$

$$\gamma(\lambda_3,\lambda_2) = (1+X)\,(-1+Y)\,\sigma + \frac{(1+X)\,(X-Y)\,(-1+Y)\,\sigma^2}{R}$$

$$+\frac{(1+X)\,(-1+Y)\,((-3+Y)\,Y + X^2\,(1+Y) - X\,(-3+2\,Y+Y^2))\,\sigma^3}{2\,R^2} + O(\frac{1}{R})^3,$$

$$\gamma(\lambda_4,\lambda_1) = -\left((-1+X+Y-X\,Y)\,\sigma\right) + \frac{(-1+X)\,(-1+Y)\,(X+Y)\,\sigma^2}{R}$$

$$+\frac{(-1+X)\,(-1+Y)\,((-3+Y)\,Y + X^2\,(1+Y) + X\,(-3+2\,Y+Y^2))\,\sigma^3}{2\,R^2} + O(\frac{1}{R})^3,$$

$$\gamma(\lambda_3,\lambda_1) = -\left((-1+X+Y-X\,Y)\,\sigma\right) + \frac{(-1+X)\,(-1+Y)\,(X+Y)\,\sigma^2}{R}$$

$$+\frac{(-1+X^2)\,(X+Y)\,(-1+Y^2)\,\sigma^3}{2\,R^2} + O(\frac{1}{R})^3,$$

$$\gamma(\lambda_4,\lambda_3) = -\frac{R^2}{\sigma} + 2\,X\,Y\,R - \left(-2+X^2+Y^2\right)\,\sigma + O(\frac{1}{R})^2.$$

By differentiating expansions in the numerator and denominator for v, we obtain expansions for the derivatives of v. Indeed,

$$A^4\frac{\partial v_n}{\partial X} = 8\left(-1+A^2\right)^2\,(-1+X)\,X\,(1+X)\,\left(-1+X^2\right)^2\,(-1+Y)^6$$

$$\left(5\,\left(-1+X^2\right)+5\,A^4\,\left(-1+X^2\right)-2\,A^2\,\left(3+5\,X^2-8\,Y^2\right)\right)\,\sigma^6 R^6+O(R^5),$$

$$A^4\frac{\partial v_d}{\partial X}=8\left(-1+A^2\right)^2\,(-1+X)\,(1+X)\,\left(-1+X^2\right)^2(-1+Y)^6$$

$$\left((1+X)^2\,(-1+5\,X)+A^4\,(-1+X)^2\,(1+5\,X)+2\,A^2\,X\,\left(5-5\,X^2+8\,Y^2\right)\right)\,\sigma^6 R^6$$

$$+O(R^5),$$

$$A^4\frac{\partial v_n}{\partial Y}=8\left(-1+A^2\right)^2(-1+X)^2\,(1+X)^2\left(-1+X^2\right)^2(-1+Y)^5$$

$$\left(3\,\left(-1+X^2\right)+3\,A^4\,\left(-1+X^2\right)-2\,A^2\,\left(3+3\,X^2+2\,Y-8\,Y^2\right)\right)\,\sigma^6 R^6+O(R^5),$$

$$A^4\frac{\partial v_d}{\partial Y}=8\left(-1+A^2\right)^2(-1+X)^2\,(1+X)^2\left(-1+X^2\right)^2(-1+Y)^5$$

$$\left(3\,A^4\,(-1+X)^2+3\,(1+X)^2+A^2\,\left(6-6\,X^2-4\,Y+16\,Y^2\right)\right)\,\sigma^6 R^6+O(R^5),$$

$$\frac{\partial v_n}{\partial X}v_d-v_n\frac{\partial v_d}{\partial X}=$$

$$\frac{8192\,p^5\,(-1+X^2)^8\,(-1+Y)^{12}\,(1+2\,p\,X-Y^2+p^2\,(X^2+Y^2))\,\sigma^{12}}{q^8}R^{12}+O(R^{11}),$$

$$\frac{\partial v_n}{\partial Y}v_d-v_n\frac{\partial v_d}{\partial Y}=\frac{16384\,p^4\,(1+p\,X)\,(-1+X^2)^8\,(-1+Y)^{12}\,Y\,\sigma^{12}}{q^6}R^{12}+O(R^{11}),$$

$$v_d^2=\frac{4096\,p^4\,(-1+X^2)^8\,(-1+Y)^{12}\,(1+2\,p\,X+Y^2+p^2\,(X^2-Y^2))^2\,\sigma^{12}}{q^8}R^{12}$$

$$+O(R^{11}).$$

The dominating terms of the above expansions do not vanish away from the z and ρ axes. Therefore, away from these locations,

$$\frac{\partial v}{\partial X}=\frac{\partial v_{Kerr}}{\partial X}+O(\frac{1}{R}),$$

$$\frac{\partial v}{\partial Y}=\frac{\partial v_{Kerr}}{\partial Y}+O(\frac{1}{R}),$$

where the derivatives of v_{Kerr} are given by

$$\frac{\partial v_{Kerr}}{\partial X} = \frac{2\,p\,\left(1+2\,p\,X-Y^2+p^2\,\left(X^2+Y^2\right)\right)}{\left(1+2\,p\,X+Y^2+p^2\,\left(X^2-Y^2\right)\right)^2},$$

$$\frac{\partial v_{Kerr}}{\partial Y} = \frac{4\,q^2\,\left(1+p\,X\right)\,Y}{\left(1+2\,p\,X+Y^2+p^2\,\left(X^2-Y^2\right)\right)^2}.$$

Property 3. Expansion around the z-axis.

We use similar technique to that above to compute values of v for small ρ. We position the origin of the $\{\rho, z\}$ coordinate system between the bodies so that $\{\lambda_i\} = \{-\sigma - \frac{R}{2}, \sigma - \frac{R}{2}, -\sigma + \frac{R}{2}, \sigma + \frac{R}{2}\}$ and expand the functions γ into series of ρ as $\rho \to 0$ and combine them according to (9.3). Because of branch points λ_i we have different expressions for z in each of the intervals $I_1 = (-\infty, -\sigma - \frac{R}{2})$, $I_2 = (-\sigma - \frac{R}{2}, \sigma - \frac{R}{2})$, $I_3 = (\sigma - \frac{R}{2}, -\sigma + \frac{R}{2})$, $I_4 = (-\sigma + \frac{R}{2}, \sigma + \frac{R}{2})$ and $I_5 = (\sigma + \frac{R}{2}, +\infty)$. The expansions are valid for any R in any compact set of z not including the branch points.

For example, for $z \in I_3$ we obtain

$$v_n = \frac{\left(-1+A^2\right)^4\,\left(R-2\,z+2\,\sigma\right)\,\left(R^2-4\,\sigma^2\right)^2\,\left(R-2\,\left(z+\sigma\right)\right)}{16\,A^4\,\left(R+2\,z-2\,\sigma\right)\,\sigma^4\,\left(R+2\,\left(z+\sigma\right)\right)} + O(\rho^2),$$

$$v_d = \left(-1+A^2\right)^2\left(R^2-4\,\sigma^2\right)^2$$

$$\frac{64\,A^2\,R^2\,\sigma^2+\left(\left(-1+A^2\right)\,R^2-4\,\left(1+A^2\right)\,R\,\sigma-4\,\left(-1+A^2\right)\,\left(z^2-\sigma^2\right)\right)^2}{16\,A^4\left(R+2\,z-2\,\sigma\right)^2\sigma^4\left(R+2\,\left(z+\sigma\right)\right)^2}+O(\rho^2),$$

$$v = \frac{\left(-1+A^2\right)^2\,\left(R+2\,z-2\,\sigma\right)\,\left(R-2\,z+2\,\sigma\right)\,\left(R-2\,\left(z+\sigma\right)\right)\,\left(R+2\,\left(z+\sigma\right)\right)}{64\,A^2\,R^2\,\sigma^2+\left(\left(-1+A^2\right)\,R^2-4\,\left(1+A^2\right)\,R\,\sigma-4\,\left(-1+A^2\right)\,\left(z^2-\sigma^2\right)\right)^2}$$

$$+O(\rho^2).$$

Similarly for z in other I_i,

$$v_{z\in I_1} = (R + 2\,z - 2\,\sigma)\;(R - 2\,z + 2\,\sigma)$$

$$\left(\left(-1 + A^2\right)^2 R^2 + 16\,A^2\,\sigma^2\right)^2\;(R - 2\;(z + \sigma))\;(R + 2\;(z + \sigma))\,/$$

$$(64\,A^2\,R^2\,\sigma^2\left(\left(-1 + A^2\right)\;R^2 + 4\,\sigma\;\left(z + A^2\,z + \sigma - A^2\,\sigma\right)\right)^2$$

$$+\left(\left(-1 + A^2\right)^2 R^4 - 4\,R^2\left(\left(-1 + A^2\right)\;z + \left(1 + A^2\right)\;\sigma\right)^2 + 64\,A^2\,\sigma^2\;\left(-z^2 + \sigma^2\right)\right)^2)$$

$$+O(\rho^2),$$

$$v_{z\in I_2} = (A^2\left(-1 + A^2\right)^2 (R - 2\,\sigma)^2\;(R + 2\,z - 2\,\sigma)\;(R - 2\,z + 2\,\sigma)$$

$$(R - 2\;(z + \sigma))\;(R + 2\;(z + \sigma)))/$$

$$(R^2\sigma^2($$

$$4\left(\left(-1 + 3\,A^2\right)\;R + 2\;\left(z + A^2\,z + \sigma - 3\,A^2\,\sigma\right)\right)^2 +$$

$$(A^2\,((-1 + A^2)\,R^3 - 4\,A^2\,R^2\,\sigma - 4\,R\,(z - \sigma)\,((-1 + A^2)\,z + 2\,A^2\,\sigma)$$

$$+8\,(-1 + A^2)\,\sigma\,(z^2 - \sigma^2))^2)/(R^2\,\sigma^2)$$

$$)) + O(\rho^2),$$

$$v_{z\in I_4} = v_{z\in I_2},$$

$$v_{z\in I_5} = v_{z\in I_1}.$$

According to these expressions v vanishes at four points of the z line and those points are located exactly where the ergosphere of the Kerr crosses the z line. Farthermore, the dominating term is clearly positive for $z \in I_i$, $i = 1, 3, 5$

and negative for $z \in I_i$, $i = 2, 4$.

Property 4. The Ernst potential does not blow up.

Indeed, recall from Chapter 4 that the Ernst potential arises from the expression

$$\overline{\mathcal{E}} = \pm \frac{\theta(U(\infty^+) + d + \frac{B}{2\pi i} + \Pi)}{\theta(U(\infty^+) + d)} \frac{\theta(-U(\infty^+) + d)}{\theta(-U(\infty^+) + d + \frac{B}{2\pi i} + \Pi)},$$

where d is a notation for $d = -U(D) - K$ and D is a nonspecial divisor. This expression is a limit value of the function χ at infinity on the upper sheet of the Riemann surface $\mathcal{L}$. The properties of χ are well studied in the Chapter 4. All the zeros and poles of χ have been found. None of them go to infinity under the degenerate procedure of Chapter 6. For the Ernst potential to blow up we would have to assume that the expression $\theta(-U(\infty^+) + d + \frac{B}{2\pi i} + \Pi)$ vanishes after the degeneration procedure is complete. But the function $\theta(-U(\lambda) + d + \frac{B}{2\pi i} + \Pi)$ does not have zeros in the neighborhood of infinity before degeneration. Also, it is analytic function of λ in the neighborhood of infinity and depends smoothly on the parameters of the degenerate procedure. Since the degenerate function is not identically zero, we can surround the infinity by a contour such that the non-degenerate function remains separated from zero on the contour while degenerating. Then the winding number $\int_C \frac{\partial \theta}{\theta} d\lambda$ computed using such a contour C is smooth under the degeneration procedure. Thus, the degenerate function cannot have zero at infinity.

Property 5. Decay at infinity.

Let us rotate the coordinate system $\{\rho, z\}$ by an arbitrary angle ϕ, where $\phi \in [0, \pi/2]$. We denote the new variables as $\{u, w\}$ and express the Ernst potential in terms of the γ's and then substitute for every γ its expansion as u goes to infinity for fixed w. The result of such computation is as follows.

$$v = 1 + O(\frac{1}{u^2}), \tag{9.6}$$

$$\Phi = \Phi_1(\phi)\frac{1}{u^4} + O(\frac{1}{u^5}), \tag{9.7}$$

where $\Phi_1(\phi)$ is a smooth function of ϕ independent from w.

Property 6. Decay at infinity.

Let us place a coordinate system $\{\rho, z\}$ such that the bodies are located on the z-axis symmetrically with respect to the ρ-axis. The centers of the event horizons have z-coordinates $-\frac{R}{2}$ and $\frac{R}{2}$. Let us introduce new variables r and ζ according to the relations $\rho = r\frac{R}{2}$ and $z = \zeta\frac{R}{2}$. A direct computation similar to those mentioned earlier shows that the function v has the following property in the new coordinate system:

$$v = \frac{v_1 R^4 + O(R^2)}{v_1 R^4 + v_2 R^3 + O(R^2)}, \tag{9.8}$$

where

$$v_1 = \left(-1 + C^2\right)^4 {F_-}^4 \left(1 - \zeta + F_-\right)^2 \left(1 + \zeta - F_+\right)^2 {F_+}^4 \left(-1 + r^2 + \zeta^2 + F_-\, F_+\right)^4,$$

$$v_2 = \left(-1 + C^2\right)^2 \left(-1 + C^4\right) {F_-}^3 \left(1 - \zeta + F_-\right)^2 \left(1 + \zeta - F_+\right)^2 {F_+}^3$$

$$\left(F_- + F_+\right) \left(-1 + r^2 + \zeta^2 + F_-\, F_+\right)^4,$$

$$F_+ = \sqrt{r^2 + (1 + \zeta)^2},$$

$$F_- = \sqrt{r^2 + (1-\zeta)^2}.$$

The above result reduces to

$$v = 1 + O\left(\frac{1}{R}\right) \tag{9.9}$$

for every point $\{r, \zeta\}$ provided that v_1 is not zero. A simple analysis shows that v_1 is equal to zero at $\{r = 0, \zeta \in [-1, 1]\}$. So, for $r = 0$ we do a separate computation and find that for $\zeta \in [-1, 1]$,

$$v = \frac{v_3 R^2 + o(R^2)}{v_3 R^2 + o(R^2)}$$

where

$$v_3 = \frac{(-1 + A^2)^2 (-1 + \zeta)^6}{16\, A^2\, \sigma^2}.$$

We next want to see that the partial derivatives of v with respect to ζ and r are uniformly bounded in every compact set of $\{r, \zeta\}$ away from the branch points ± 1 and including the ζ-axis. For this we differentiate expansions for the numerator and the denominator and note that they are polynomial expressions and the sum of the dominating terms $v_1 R^4$ and $v_3 R^2$ is separated from zero both in general area and around the ζ axis. Thus, the numerator of the derivative is bounded and the denominator is separated from zero.

We conclude that v converges to 1 everywhere in the neighborhood of the ζ line away from ± 1. If this were not so then there is a sequence of points on the $\{r, \zeta\}$ plane in a neighborhood of the ζ axis such that $v(r_n, \zeta_n, R_n)$ remains separated from 1. Thus the partial derivatives of $v(r, \zeta)$ can not stay bounded uniformly with respect to R.

Property 7. Decay at infinity.

It is another direct computation to verify that $v \to 1$ if $\rho^2 + z^2 \to \infty$ and $R \to \infty$. Precisely, suppose $\rho = Fz$, $F = \text{const}$ and $R \to \infty$, $z \to \infty$, $R/z \to 0$, then we have

$$v = \frac{f^2 + O(\text{smaller terms})}{f^2 + O(\text{smaller terms})},$$

$$f = \frac{F^6\,(1+F^2)^2\,z^{12}}{R^{12}} - \frac{16\,(1+F)^2\,(1+F^2)\left(-1+\sqrt{1+F^2}\right)\,z^8}{R^6\,\sigma^2}$$
$$+\frac{8\,(1+F)^2\,(1+F^2)\left(-1+\sqrt{1+F^2}\right)\,z^8}{C^2\,R^6\,\sigma^2}+$$
$$\frac{8\,C^2\,(1+F)^2\,(1+F^2)\left(-1+\sqrt{1+F^2}\right)\,z^8}{R^6\,\sigma^2}.$$

Suppose that the λ_j's above are located symmetrically with respect to the line $Re\lambda = 0$. We now prove that the Ernst potential obtained this way has the property

$$\mathcal{E}(\rho, z) = \mathcal{E}(\rho, -z),$$

which is physically reasonable as the bodies rotate in opposite directions. Ultimately this symmetry is needed to prove that $A(\rho = 0, z) = 0$ (see (7.4)). Before degeneration, the RHP is shown in Figure 9.3. The matrix valued function Ψ constructed in Chapter 4 and considered as a function defined in a complex plane satisfies the following set of conditions:

1. $\Psi_+ = \Psi_- i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on every non-moving cut.
2. $\Psi_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi_- \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the moving cut.

3. $\Psi(\infty) = \begin{pmatrix} \overline{\mathcal{E}} & 1 \\ \mathcal{E} & -1 \end{pmatrix}$.

4. $\Psi = \hat{\Psi}(s) \begin{pmatrix} 1 & 0 \\ 0 & s^{1/2} \end{pmatrix} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}$ around the non-moving cut branch points. Here s is the local parameter and $\hat{\Psi}$ is some locally analytic and invertible matrix valued function.

5. $\Psi = \hat{\Psi}(s) \begin{pmatrix} s^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ around the points in the divisor D. The divisor D has to be real and we choose it so that $D = -D$.

6. $\Psi = \hat{\Psi}(s) \begin{pmatrix} e^{\alpha_k/s} & 0 \\ 0 & e^{-\alpha_k/s} \end{pmatrix}$ around each of the four points $\lambda = p_k$ for $k = 1, ..., 4$. The quantities $\{p_k\}$ have to be real and we choose them so that $p_1 = -p_4$ and $p_2 = -p_3$. Also the α_k's have to be real and connected to C_k by relations (6.7) and (6.8).

Perform the transformation $z \to -z$. This shifts the moving cut to a different location but preserves its position relative to the non-moving cuts. The conditions (1)-(6) are preserved except for (3) where the $\mathcal{E}(\rho, z)$ changes to $\mathcal{E}(\rho, -z)$.

We now perform the transformation $\lambda \to -\lambda$ to return the moving cut to its original position and use the uniqueness result of Chapter 3. The conditions (1) and (2) are such that switching $+$ to $-$ does not change them. Condition (3) is not affected by the transformation. In conditions (4) and (5), $\hat{\Psi}$ is changed but in an allowed fashion. Now consider condition (6). Around p_2 and $p_3 = -p_2$ (see (6.6)),

$$W(\lambda) = W(p_2 + s) = \frac{\alpha}{s}(1 + O(1)),$$

$$W(\lambda) = W(-p_2 + s) = -\frac{\alpha}{s}(1 + O(1)).$$

Hence,

$$W(-\lambda) = W(-(p_2 + s)) = W(-p_2 - s) = \frac{\alpha}{s}(1 + O(1)).$$

So, $W(-\lambda) = \frac{\alpha}{s}(1 + O(1))$ around p_2. Hence, the transformation $\lambda \to -\lambda$ preserves the parameter α.

We conclude that the chain of transformations $z \to -z$ and $\lambda \to -\lambda$ do not change the RHP and hence the potential $\mathcal{E}$ must have the desired property

$$\mathcal{E}(\rho, z) = \mathcal{E}(\rho, -z) \tag{9.10}$$

for the non-degenerate RHP. Since it is possible to perform the degeneration while maintaining the symmetries, the degenerate Ernst potential also has the desired symmetry.

Of course, this property may also be verified by direct computation.

One may ask the following question: why doesn't the same argument as above prove that property (9.10) also holds for the Kerr solution which, in fact, does not have such a property (see 5.4)? To answer this question let us examine the positioning of the singularities of W. According to (6.7) and (6.8)

$$\alpha_s = \omega_s(p_s - b_s),$$

$$C_k = e^{4\omega_k},$$

where b_s is the one of two non-moving branch points which is closer to the moving cut. For the symmetry needed in the argument we must position the singularities p_i of W symmetrically as in Figure 9.4.

Under the limiting procedure, we have to move p_i to b_i at the same rate for both $i = 1$ and $i = 2$ because this rate is given by the absolute value of the parameter ω_i which is the same for $i = 1$ and $i = 2$. Clearly we cannot do that

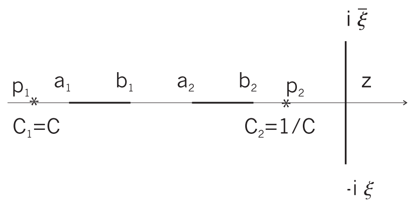

Figure 9.4: Symmetric positioning of singularities of W.

if we want to keep the picture symmetrical.

We are now in a position to prove that the function A defined according to (1.6)-(1.7) with $\mathcal{E} = v + i\Phi$ defined by the RHP on Figure 9.3 has the property $A = 0$ everywhere on the z-axis outside of the event horizons. One checks first that the functions v and Φ are such that the RHS of (7.2)-(7.3) insure that A may be made zero at infinity on the half plane $\{\rho > 0, z\}$ by a choice of an integration constant. This means that A should vanish as $\rho^2 + z^2 \to \infty$.

As we noted before, the function Φ vanishes as fourth power as $\rho^2+z^2 \to \infty$. This is not sufficient however to conclude that the derivatives Φ vanish. Thus, we repeat the computation that leads to **property** 5 above with complex variables u and w. We have the following result:

$$\Phi(u + Ia, w) = O(\frac{1}{u^3}), \tag{9.11}$$

$$\Phi(u, w + Ia) = O(\frac{1}{u^3}). \tag{9.12}$$

For technical reasons it is difficult to directly check if Φ with complex argument vanishes as a fourth power. But the third power is sufficient for our purposes. We invoke the Cauchy formula:

$$\frac{\partial \Phi}{\partial u}(u, w) = \frac{-1}{2\pi i} \int_{C_a} \frac{\Phi(z, w)}{(z-u)^2} dz,$$

where C_a is a circle of radius a centered around the u in the complex plane. Because of the expansions (9.11)-(9.12) we conclude that the derivative also vanishes as a third power. We argue similarly for the dervative in the w variable.

Therefore,

$$\frac{\partial A}{\partial u} = O(\frac{1}{u^2}), \tag{9.13}$$

$$\frac{\partial A}{\partial w} = O(\frac{1}{u^2}). \tag{9.14}$$

We now can make A to be zero everywhere at infinity by a choice of a single integration constant. Indeed, A is constant on the z-axis away from the event horizons. So, we choose the constant of integration so that $A = 0$ on the positive unbounded part of the z-axis. Every point P in the $\{\rho, z\}$ plane suffciently far away may be connected to the part of the z-axis by an arc centered at the origin. The value of the function A at P is an integral along the arc of the derivative along the direction tangent to the arc, which is the same as the derivative with respect to w above. As P moves away to infinity, the length of the arc grows like the first power of distance to the origin but the derivative decays like the second power. Therefore, by the mean value theorem, A vanishes as P goes to infinity.

We now prove that $A = 0$ on the part of the z axis between the bodies. Because of the symmetry (9.10) we have $\Phi(\rho, z) = \Phi(\rho, -z)$. So, $\Phi_z(\rho, z) = -\Phi_z(\rho, -z)$. Therefore, $\Phi_z(\rho, 0) = 0$. Using (7.2)-(7.3) we conclude that $A_\rho(\rho, 0) = 0$. Hence $\{(\rho, z), z = 0\}$ is a level curve of A. As $A(\rho = \infty, z = 0) = 0$, we conclude that $A(\rho = 0, z = 0) = 0$. Then $A(\rho = 0, z) = 0$ on the entire interval between the event horizons because A is constant on the line $\rho = 0$. In the above argument we used the fact that the derivatives of A are not singular as ρ goes to 0. This property may be verified by expanding the functions Φ and v into powers of ρ for small ρ.

In Chapter 11 we will see that the property $A(\rho = 0, z) = 0$ is sufficient to prove that the solution we obtained is the only physical solution for the configuration of two rotating black holes.

So far we obtained all the interesting solutions by degenerating a RHP. We may try to consider the degenerate problem directly. As we discuss in Chapter 10 such a degenerate RHP does not exist.

The next problem in the procedure is the actual explicit integration of the equations (1.8)-(1.9) with known v and Φ. As we mentioned before, one can see from (6.2)-(6.5) and (6.1) that the functions v and Φ in (1.8)-(1.9) are of the form (9.5). However, F is a complicated rational function of its arguments and we cannot hope to compute the indefinite integrals of the right hand sides of (1.8)-(1.9) explicitly with such v and Φ.

We are going to use the prolate spheroidal coordinates $\{X, Y\}$ presented earlier.

$$X = \frac{1}{2\sigma}(r_+ + r_-),$$

$$Y = \frac{1}{2\sigma}(r_+ - r_-),$$

$$r_+^2 = \rho^2 + (z + \sigma)^2,$$

$$r_-^2 = \rho^2 + (z - \sigma)^2.$$

After expansion in powers of $1/R$, all the expressions for β became rational functions of X and Y and so did the Ernst potential. This allows us to compute expressions for indefinite integrals of the RHS of (1.8)-(1.9). Because (1.8)-(1.9) are compatible, this is sufficient to compute the function K.

What follows are expressions for the functions K, the force F and the distance between the bodies $\mathcal{D} = \int_{\sigma}^{R-\sigma} \sqrt{ds^2}$, where $\sqrt{ds^2}$ is the infinitesimal metric length. It is a formidable task to compute the terms by hand, and we have again used software for analytical manipulation. To further simplify the computations we **assume that the rotation parameter a is small** compared to the mass m.

We obtain

$$\mathcal{D} = R + (\frac{9\,a^4}{8\,m^3} + \frac{5\,a^2}{2\,m} - m - 2\,m\,\log(m) - 2\,m\,\log(\frac{1}{R}) + O(a^5))$$
$$+\frac{2\,a^2 + \frac{3\,a^4}{4\,m^2} - 7\,m^2 - 2\,m^2\,\log(m) - 2\,m^2\,\log(\frac{1}{R}) + O(a^5)}{R} + O(\frac{\log(R)}{R^2}) \quad (9.15)$$

$$F = \frac{m^2}{R^2} + \frac{-4\,a^2\,m^2 + 4\,m^4 + O(a^8)}{R^4} + O\left(\frac{1}{R^5}\right). \quad (9.16)$$

So far we are assuming R to be large. This is an essential assumption needed in the computations and the arguments. The assumption of smallness of a, however, is not essential and is only needed to control the size of the computations.

Again, the quantity $\mathcal{D}$ is the observed distance between the bodies and R is the difference of the z coordinates of the centers of the bodies. We are going to compare forces acting between two bodies in the case when they are non-rotating and when they are rotating in opposite directions. In both cases we place the bodies the same observed distance $\mathcal{D}$ apart. As explained in the Chapter 8 the parameters m and ma are the mass and angular momenta of each of the bodies when they are separated. So, we have the following quantities:

$$\mathcal{D} = R + \alpha + \beta\log(R) + \frac{\gamma}{R} + \delta\frac{\log(R)}{R} + O(\frac{\log(R)}{R^2}), \quad (9.17)$$

$$\mathcal{D} = R_0 + \alpha_0 + \beta \log(R) + \frac{\gamma_0}{R_0} + \delta \frac{\log(R_0)}{R_0} + O(\frac{\log(R_0)}{R_0^2}),$$

$$F = \frac{m^2}{R^2} + \frac{f}{R^4} + O\left(\frac{1}{R^5}\right),$$

$$F_0 = \frac{m^2}{{R_0}^2} + \frac{f_0}{{R_0}^4} + O\left(\frac{1}{{R_0}^5}\right),$$

where F is the force acting between rotating bodies. Here the subscript 0 refers to the case with no rotation. From above,

$$\alpha = \frac{9\,a^4}{8\,m^3} + \frac{5\,a^2}{2\,m} - m - 2\,m\,\log(m) + O(a^5),$$

$$\beta = 2\,m,$$

$$\gamma = 2\,a^2 + \frac{3\,a^4}{4\,m^2} - 7\,m^2 - 2\,m^2\,\log(m) + O(a^5),$$

$$\delta = 2\,m^2,$$

$$f = -4\,a^2\,m^2 + 4\,m^4 + O(a^8),$$

$$\alpha_0 = -m - 2\,m\,\log(m),$$

$$\gamma_0 = -7\,m^2 - 2\,m^2\,\log(m),$$

$$f_0 = 4\,m^4.$$

So far the force has been computed in terms of R, which is not an invariant quantity. Using (9.15) and (9.17) we express $\mathcal{D}$ in terms of R and substitute it into the expressions for the force (9.16). We obtain the difference $F - F_0$ in terms of $\mathcal{D}$, m and a:

$$
\begin{aligned}
F &= \frac{m^2}{\mathcal{D}^2} - \frac{2\,m^2\,(-\alpha-\beta\,\log(\mathcal{D}))}{\mathcal{D}^3} + \\
&\frac{f + \frac{m^2\,\left(6\,(-\alpha-\beta\,\log(\mathcal{D}))^2 - 4\,\left(\alpha\,\beta-\gamma+\left(\beta^2-\delta\right)\,\log(\mathcal{D})\right)\right)}{2}}{\mathcal{D}^4} + O(\frac{1}{\mathcal{D}})^5 \\
&= \frac{m^2}{\mathcal{D}^2} + \\
&\frac{1}{\mathcal{D}^3}\left(\frac{9\,a^4}{4\,m} + 5\,a^2\,m - 2\,m^3 + 4\,m^3\,\log(\mathcal{D}) - 4\,m^3\,\log(m) + O(a^6)\right) + \\
&\frac{1}{\mathcal{D}^4}(9\,a^4 - 25\,a^2\,m^2 - 3\,m^4 + 12\,m^4\log(\mathcal{D})^2 + \\
&\left(\frac{-27\,a^4}{2} - 30\,a^2\,m^2 + 16\,m^4\right)\,\log(m) + 12\,m^4\log(m)^2 + \log(\mathcal{D}) \\
&\left(\frac{27\,a^4}{2} + 30\,a^2\,m^2 - 16\,m^4 - 24\,m^4\,\log(m)\right) + O(a^6)) + O\left(\frac{1}{\mathcal{D}^5}\right),
\end{aligned}
$$

$$
\begin{aligned}
F - F_0 &= \frac{\frac{9\,a^4}{4\,m} + 5\,a^2\,m + O(a^6)}{\mathcal{D}^3} + \\
&(9\,a^4 - 25\,a^2\,m^2 + \frac{27\,a^4\,\log(\mathcal{D})}{2} + 30\,a^2\,m^2\,\log(\mathcal{D}) \\
&-\frac{27\,a^4\,\log(m)}{2} - 30\,a^2\,m^2\,\log(m) + O(a^6))/(\mathcal{D}^4) + \\
&O(\frac{1}{\mathcal{D}})^5 .
\end{aligned}
\tag{9.18}
$$

Observe that $m^2/\mathcal{D}^2$ is the Newtonian part of the force, as previously noted by [Weyl]. We see however, and this is the **main result** of this thesis, that the force between two bodies **increases** by

$$
\frac{\frac{9\,a^4}{4\,m} + 5\,a^2\,m + O(a^6)}{D^3}, \tag{9.19}
$$

for large $\mathcal{D}$ and $a \neq 0$. Note again that in (9.19) a is assumed to be small compared to m.

Formula (9.18) cannot be compared directly with the analogous formula (???) in [Dietz]. The reason is the following: in (9.19) we are comparing bodies with equal intrinsic masses and angular momenta whereas in [Dietz] the authors compare bodies with equal Komar masses and angular momenta.

CHAPTER 10

Real potential case.

Unfortunately, The solution of RHP degenerates in the limit. This means that we were not able to solve for our desired Ernst potential directly. One might ask: is it possible to find some other RHP, of the type considered in Chapter 3, that would yield the two rotating bodies solution of EVE directly? We show that this is not possible.

Recall that the solution of the RHP problem we used goes to zero everywhere except for infinity under the limiting procedure of Chapter 6. Indeed, we have (4.3)

$$\Psi = \begin{pmatrix} \chi & \chi^* \\ \tilde{\chi} & -\tilde{\chi}^* \end{pmatrix},$$

with χ given by (4.4)

$$\chi(\lambda) = C\frac{\theta(U(\lambda) - U(D) - K + \frac{B_w}{2\pi i} + \{-\frac{1}{4}\})}{\theta(U(\lambda) - U(D) - K)}e^W,$$

where C is chosen according to (4.2). Thus,

$$\chi(\lambda) = \frac{\theta(-U(\infty^+) - U(D) - K)}{\theta(U(\infty^+) - U(D) - K + \frac{B_w}{2\pi i} + \{-\frac{1}{4}\})}$$

$$\frac{\theta(U(\lambda) - U(D) - K + \frac{B_w}{2\pi i} + \{-\frac{1}{4}\})}{\theta(U(\lambda) - U(D) - K)}e^{W - W(\infty^+)}.$$

Suppose we have one body. We perform the limiting procedure in such a way that $D \to \{\infty^+, \infty^-\}$. So, $\theta(U(\infty^+) - U(D) - K) \to 0$ and χ goes to zero everywhere, except for infinity, where $\theta(U(\infty^+) - U(D) - K)$ goes to zero as well. Thus, we can not obtain a RHP for a limiting Ernst potential just by taking a limit of the our existing RHP.

Let us consider the case of a real Ernst potential. What happens to the Lax couple? According to Chapter 2,

$$\Psi_\xi = a\,(e + f\sigma)\,\Psi,$$

$$\Psi_{\overline{\xi}} = \sigma\overline{a}\left(\sigma + \frac{1}{f}e\right)\Psi,$$

$$\Psi(\infty^+) = \begin{pmatrix} \overline{\mathcal{E}} & 1 \\ \mathcal{E} & -1 \end{pmatrix},$$

$$f = \sqrt{\frac{\lambda - i\overline{\xi}}{\lambda + i\xi}},$$

$$a = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$A = \frac{\overline{\mathcal{E}}_\xi}{\overline{\mathcal{E}} + \mathcal{E}},$$

$$B = \frac{\mathcal{E}_\xi}{\overline{\mathcal{E}} + \mathcal{E}}.$$

If $\mathcal{E}$ is real then $A = B$. Now

$$\Psi_\xi = A\,(e + f\sigma)\,\Psi,$$

$$\Psi_{\overline{\xi}} = \overline{A}\sigma\left(\sigma + \frac{1}{f}e\right)\Psi = \overline{A}\left(e + \frac{1}{f}\sigma\right)\Psi$$

Let

$$\gamma = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

$$\Phi = \gamma \Psi,$$

then

$$\Phi_{\xi} = A\left(e + f\gamma\sigma\gamma^{-1}\right)\Phi,$$

$$\Phi_{\overline{\xi}} = \overline{A}\left(e + \frac{1}{f}\gamma\sigma\gamma^{-1}\right)\Phi.$$

Observe that

$$\gamma^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

$$\gamma\sigma\gamma^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Denote

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \epsilon,$$

then

$$\Phi_{\xi} = A\left(e + f\epsilon\right)\Phi,$$

$$\Phi_{\overline{\xi}} = \overline{A}\left(e + \frac{1}{f}\epsilon\right)\Phi,$$

$$\Phi(\infty^{+}) = \begin{pmatrix} 0 & 2 \\ 2\mathcal{E} & 0 \end{pmatrix}.$$

Since

$$\Psi = \begin{pmatrix} \chi & \chi^{*} \\ \tilde{\chi} & -\tilde{\chi}^{*} \end{pmatrix}$$

then

$$\Phi = \begin{pmatrix} \chi - \tilde{\chi} & \chi^* + \tilde{\chi}^* \\ \chi + \tilde{\chi} & \chi^* - \tilde{\chi}^* \end{pmatrix}.$$

If χ is real these would imply

$$\chi_\xi = A\,(1+f)\,\chi,$$

$$\chi_{\overline{\xi}} = \overline{A}\left(1+\frac{1}{f}\right)\chi,$$

$$\chi(\infty^+) = \mathcal{E}.$$

These equations may be solved up to quadrature. In the case of the Schwartzchild potential

$$\mathcal{E} = \frac{X-1}{X+1}$$

(see (5.3)), we have:

$$\chi = \sqrt{G(\lambda, \xi, \overline{\xi})},$$

$$G(\lambda,\xi,\overline{\xi}) = -\frac{\left(-\sqrt{m+i\,\overline{\xi}}+\sqrt{m-i\,\xi}\right)\,\left(-\sqrt{m-i\,\overline{\xi}}+\sqrt{m+i\,\xi}\right)}{\left(\sqrt{m+i\,\overline{\xi}}+\sqrt{m-i\,\xi}\right)\,\left(\sqrt{m-i\,\overline{\xi}}+\sqrt{m+i\,\xi}\right)}$$

$$\frac{2\sqrt{m-i\,\overline{\xi}}\,\left(\overline{\xi}+i\,\lambda\right)\,\sqrt{m+i\,\xi}+\frac{\sqrt{-i\,\overline{\xi}+\lambda}\left(-2\,i\,m\,\lambda-\overline{\xi}\,(m+\lambda)+\left(m-2\,i\,\overline{\xi}+\lambda\right)\xi\right)}{\sqrt{\lambda+i\,\xi}}}{-2\sqrt{m+i\,\overline{\xi}}\,\left(\overline{\xi}+i\,\lambda\right)\,\sqrt{m-i\,\xi}+\frac{\sqrt{-i\,\overline{\xi}+\lambda}\left(-2\,i\,m\,\lambda+\overline{\xi}\,(-m+\lambda)+\left(m+2\,i\,\overline{\xi}-\lambda\right)\xi\right)}{\sqrt{\lambda+i\,\xi}}}.$$

Now one sees by direct computation that the numerator and the denominator have two zeros each and the position of the zeros depends on ξ and $\overline{\xi}$. Such χ may not be obtained from the RHP presented in Chapter 3.

CHAPTER 11

Uniqueness.

We have seen that a solution with physical meaning should have $A = 0$ on the z-axis. In this chapter we will outline a proof that such a solution is the only physical solution for two rotating black holes with the specified parameters. We will be using results in [Wein]. As noted in the Introduction, our proof of uniqueness is not completely rigorous, and rests on some plausible assumptions about the nature of the solution.

First, we introduce another version of the Ernst potential. Instead of taking $Re(\mathcal{E}) = -g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})$ we take $Re(\mathcal{E}) = g(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi})$. This is possible, as may be seen from the discussion following (1.1). In the case of the Kerr solution, the function $e^{2U} = Re\mathcal{E}$ vanishes outside of the event horizon. We choose the different Ernst potential because we do not want $Re\mathcal{E}$ to have this property.

Following the notation in [Wein] we state the result of a computation similar to the one in Chapter 1. One solves the equations

$$\partial^A \left(\rho \partial_A X\right) = \frac{\rho}{X}\left(|\partial X|^2 - |\partial Y|^2\right), \tag{11.1}$$

$$\partial^A \left(\rho \partial_A Y\right) = \frac{2\rho}{X}\partial^A X \partial_A Y, \tag{11.2}$$

with undetermined functions $X = X(\rho, z)$, $Y = Y(\rho, z)$ and $A \in \{\rho, z\}$ and recovers functions $\Omega = \Omega(\rho, z)$ and $\omega = \omega(\rho, z)$ according to the relations

$$d\Omega = -\frac{\rho}{X^2} * dY, \tag{11.3}$$

$$*d\rho = dz, *dz = -d\rho,$$

$$d\omega = \frac{\rho}{4X^2} Re\left(\left((\partial X)^2 + (\partial Y)^2\right) d\zeta\right), \tag{11.4}$$

$$d\zeta = d\rho + i dz,$$

$$\partial = \partial_\rho - i\partial_z.$$

Then the axially symmetric vacuum space time is given by

$$ds^2 = -V dt^2 + 2W d\phi dt + X d\phi^2 + e^{2\lambda}\left(d\rho^2 + dz^2\right), \tag{11.5}$$

where $W = X\Omega$, $V = X^{-1}(\rho^2 - W^2)$ and $e^{2\lambda} = X^{-1}e^{2\omega}$.

Note that equations (11.1) and (11.2) are equivalent to the Ernst equation (1.3) for a new Ernst potential $X + iY$. The relations (11.3) and (11.4) reduce to the following:

$$\partial_\rho \Omega = \frac{\rho}{X^2}\partial_z Y,$$

$$\partial_z \Omega = -\frac{\rho}{X^2}\partial_\rho Y,$$

$$\partial_\rho \omega = \frac{\rho}{4X^2}\left((\partial_\rho X)^2 - (\partial_z X)^2 + (\partial_\rho Y)^2 - (\partial_z Y)^2\right),$$

$$\partial_z \omega = \frac{\rho}{2X^2}\left(\partial_\rho X \partial_z X + \partial_\rho Y \partial_z Y\right).$$

It is remarkable that

$$I(K) = 2\pi \int_C dY,$$

where C is a contour in Chapter 8.

Proposition 1.

Suppose $F_1 = \{X_1, Y_1\}$ *and* $F_2 = \{X_2, Y_2\}$ *are two solutions of the equations (11.1)-(11.2), then the function*

$$d(\rho, z) = dist(F_1, F_2) = f\left(\frac{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}{2X_1X_2}\right).$$

$$f(x) = x^2,$$

is a subharmonic function in 3d space.

The **proof** of this proposition is a direct verification. Indeed, write

$$d(\rho, z) = f(D),$$

where $D = \frac{(X_1-X_2)^2+(Y_1-Y_2)^2}{2X_1X_2}$. We compute the Laplacian

$$\Delta d = \frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho d\right) + \partial_z^2 d =$$

$$= \partial^2 f((\partial_\rho D)^2 + (\partial_z D)^2) + \partial f \Delta D.$$

So, given that $f' \geq 0$ and $f'' \geq 0$ it is enough to show that $\Delta D \geq 0$. One does a direct computation of ΔD in terms of X and Y and its derivatives and eliminates second derivatives of X and Y using the relations (11.1)-(11.2). Then the expression for the Laplacian becomes a quadratic form over the first derivatives. The matrix of the the quadratic form happens to be nonnegative definite. So, $\Delta d \geq 0$.

Proposition 2.

Let $B = B_R \subset R^n$ *be the ball of radius* R *centered at the origin, and let* Σ *be a smooth closed submanifold of* R^n *of codimension at least 2. Suppose that* $u \in$

$C^\infty(\overline{B}\backslash\Sigma)$ *satisfies* $\Delta u \geq 0$, *and* $0 \leq u \leq 1$ *in* $B\backslash\Sigma$. *Then* $\sup_{B\backslash\Sigma} u \leq \sup_{\partial B\backslash\Sigma} u$. The proof may be found in [Wein1], p.1413.

Suppose we have two bodies with intrinsic masses (see Chapter 8) m_1, m_2 and intrinsic angular momentas m_1a_1, m_2a_2 which are separated by a (coordinate) distance R.

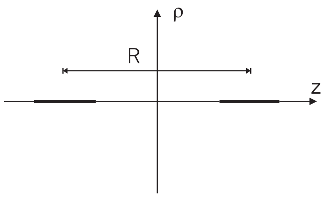

Figure 11.1: Event horizons in ρ,z plane.

Let us denote by $F_1 = \{X_1, Y_1\}$ the Kerr solution with parameters m_1, a_1 centered at the origin, $F_2 = \{X_2, Y_2\}$ the Kerr solution with parameters m_2, a_2 centered at the point $(\rho = 0, z = R)$, $F_3 = \{X_3, Y_3\}$ the Kerr solution with parameters $m_1 + m_2$, $a_1 + a_2$ centered at the origin. Let σ_i, for $i = 1, 2$ be given by the relation $\sigma_i = \sqrt{m_i^2 - a_i^2}$. Let Σ denote the union of the disjoint intervals $I_1 = [-\sigma_1, \sigma_1]$ and $I_2 = [-\sigma_2 + R, \sigma_2 + R]$ on the z axis, where R is sufficiently large.

The Problem. *Choose five numbers* m_1, m_2, a_1, a_2, $R > \sigma_1 + \sigma_2$ *and find a pair of smooth functions* $X = X(\rho, z)$, $Y = Y(\rho, z)$ *for* $(\rho, z) \in R^2_+ = \{\rho \geq 0\}$, *where* $X \geq 0$ *in* R^2_+ *and* X *is allowed to vanish only at the z-axis, satisfying the equations (11.1)-(11.2) in* $\{(\rho, z), \rho > 0\}$ *with the boundary conditions*

1. $dist(F_1, F)$ *is bounded in some neighborhood of* I_1,

2. $dist(F_2, F)$ *is bounded in some neighborhood of* I_2,

3. $dist(F_3, F) \to 0$ *as* (ρ, z) *runs away from both of the bodies. Here* $F = \{X, Y\}$.

Theorem. *The solution of the above problem is unique.*

The proof is a standard and direct application of Propositions 1 and 2 to the distance $dist(.,.)$ between two solutions of the problem. The ball B mentioned in Proposition 2 is in $\{\rho, z, \phi\}$ space. Here, Σ is the z-line. We assume the existence of two solutions F_1, F_2 and apply Proposition 2 to the function $u = dist(F_1, F_2)$ to conclude that $u = 0$.

Let $\mathcal{E} = v + i\Phi$ be the Ernst potential for two equal bodies rotating in opposite directions as discussed in Chapter 9. It gives rise to $\{X, Y\}$ satisfying (11.1),(11.2) in the following sence. Set

$$X = \frac{1}{v}\rho^2 - vA^2 \tag{11.6}$$

and let Y solve

$$\partial_\rho Y = -2A\frac{\rho}{v}\partial_z v + \left(\frac{\rho^2}{v^2} + A^2\right)\partial_\rho \Phi, \tag{11.7}$$

$$\partial_z Y = 2A\left(-1 + \frac{\rho}{v}\partial_\rho v\right) + \left(\frac{\rho^2}{v^2} + A^2\right)\partial_z \Phi. \tag{11.8}$$

These two relations defining Y are suggested by the computations of the Komar integral $I(K)$ (see Chapter 8). The definition of X follows by comparing the space time metric in the two forms (1.12) and (11.5). The compatibility of (11.7) and (11.8) follows directly from equations (1.3), (1.6) and (1.7). To complete the argument, we need to show that the couple $F = \{X, Y\}$ produced according to the above relations satisfies the conditions in **The Problem**.

The first issue we will address is the smoothness of $\{X, Y\}$. By properties 1 and 4 of Chapter 9, v and Φ are infinitely smooth. However, expression (11.6) is such that X may or may not be bounded if v vanishes. So, we start by studying the function v for sufficiently large R.

Initially, we show that the neighborhood of the ergosphere of each of the bodies is the only place where v may vanish. Consider v in $\{r, \zeta\}$ coordinates, introduced in **property 6** of Chapter 9. We choose a small number $\epsilon > 0$ and a large number L which is to be determined later and consider a ball B_L centered at the origin of $\{r, \zeta\}$ with radius L. By **property 1**, the function v is real analytic in B_L, except for a finite number of points at the event horizons of the bodies where the quantities

$$\sqrt{(z-\lambda_1)^2+\rho^2}, ..., \sqrt{(z-\lambda_4)^2+\rho^2},$$

vanish. We cover the event horizons by small compact sets B_- and B_+ in the $\{r, \zeta\}$ plane in such a way that every square root is bigger than a certain number ϵ_0 outside of B_- and B_+.

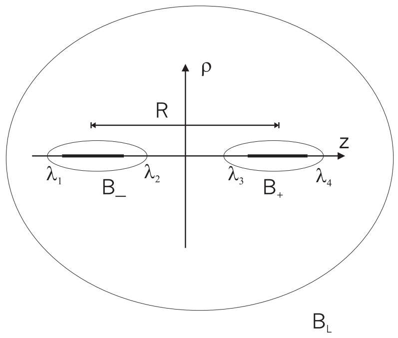

Figure 11.2: B_L, B_+ and B_-.

We also want B_- and B_+ to shrink when $R \to \infty$ in such a way that images of B_- and B_+ in the $\{\rho, z\}$ plane would have bounded diameters. This is possible for the following reason. Around B_+ we require that

$$\sqrt{(z-\frac{R}{2}\pm\sigma)^2+\rho^2}\geq\epsilon_0,$$

so it is enough to have

$$(z-\frac{R}{2}\pm\sigma)^2\geq\epsilon_0^2, \tag{11.9}$$

or

$$\rho\geq\epsilon_0. \tag{11.10}$$

Since $z=\zeta\frac{R}{2}$, the relation (11.9) is equivalent to

$$\left((\zeta-1)\frac{R}{2}\pm\sigma\right)^2\geq\epsilon_0^2. \tag{11.11}$$

Set $\zeta-1=\frac{\delta}{R}$ to be a boundary of B_+, for some δ. Then (11.11) will be satisfied if δ is chosen in agreement with

$$(\frac{\delta}{2}\pm\sigma)^2\geq\epsilon_0^2,$$

which is possible. Relation (11.10) clearly gives a shrinking boundary in the $\{r,\zeta\}$ plane.

In the $\{r,\zeta\}$ plane the function v is real analytic in $B\equiv B_L/\{B_-\cup B_+\}$. According to **property 6** , it converges pointwise to 1 in B as $R\to\infty$. Derivatives of v in $\{r,\zeta\}$ are also real analytic. Since, B is a compact set in $\{r,\zeta\}$, the derivatives of $v(R,r,\zeta)$ are bounded. Thus, v is uniformly continuous in B as $R\to\infty$. So, we have $v(R,r,\zeta)$ converging pointwise to a continuous limit and is uniformly continuous on B as $R\to\infty$. Therefore, the convergence is uniform.

We take R sufficiently large to insure $|v-1| \leq \epsilon$ in B.

We now return to the $\{\rho, z\}$ plane. The sets B_L and $B_{\pm}$ have compact images in the $\{\rho, z\}$ plane for each R. The uniform result $|v-1| \leq \epsilon$ in B is preserved in $\{\rho, z\}$ plane.

Let us concentrate on B_+ and shift coordinate system in such a way that the origin is located in a middle of the + event horizon. The set B_+ is chosen in such a way that all images of B_+ for different R may be covered by a compact set in shifted $\{\rho, z\}$ plane. We denote this set by B_+^0. Inside of B_+^0, v is a smooth function, converging to the corresponding Kerr solution $v_{Kerr,+}$ by **property 2** as $R \to \infty$. The convergence is uniform in every compact set of $\{\rho, z\}$ away from the z-line. So, by further increasing R, we have $|v-1| \leq \epsilon$ in B and $|v - v_{Kerr,\pm}| \leq \epsilon$ in each of $B_{\pm}^0$. The parameter L remains free in the above argument.

We now show that v can not vanish outside of B_L for L sufficiently large. Assume the contrary. Take any L and $R(L)$ sufficiently large s.t. v is close to 1 inside of B_L for all R exceeding $R(L)$. We assume there always exists a point p_L in the $\{\rho, z\}$ plane outside of B_L and $R_L \geq R(L)$ for which $v(R_L, p_L) = 0$. This is not possible by **property 7**.

This concludes our proof that v can vanish only around the ergospheres.

Next, we study the form of the ergosphere of Kerr's solution given by equation (11.18). As we mentioned in Chapter 5, the prolate spheroidal coordinates X,Y are connected to ρ,z coordinates by the relations

$$\rho^2 = \sigma^2(X^2-1)(1-Y^2),$$

$$z = \sigma XY.$$

We produce a formal solution of these three equations ((11.18) and the two above) in the form

$$\rho^2 = \frac{\sigma^2}{q^2}\left(1 - X^2(p^2 - q^2) - q^2\frac{z^2}{\sigma^2} - q^2\right), \tag{11.12}$$

where X^2 may be expressed as

$$X^2 = \frac{1 \pm \sqrt{1 - 4p^2q^2z^2/\sigma^2}}{2p^2}. \tag{11.13}$$

It follows from the definition of X in Chapter 5 that X has to be larger then 1. For values of $p \leq 1/2$, which corresponds to slow rotation, only one solution $\rho^2 = \rho^2(z)$ given by the last two relations satisfies the requirement $X^2 > 1$. We conclude that the ergosphere is a smooth curve which can be expressed as $\rho^2 = \rho^2(z)$. It intersects the z-line at points $-\sigma$ and σ and has the symmetry $z \to -z$. Furthermore, in $\{\rho^2, z\}$ coordinates the ergosphere has a finite slope as may be established by differentiating (11.12) and (11.13).

According to the **the property 2** of Chapter 9, v has an expansion

$$v = v_{Kerr} + \frac{v_1}{R} + O(\frac{1}{R^2}), \tag{11.14}$$

where v_{Kerr} is a real part of the Ernst potential for the Kerr solution and v_1 vanishes at the ergosphere of the Kerr solution. Here, the $\{\rho, z\}$ coordinate system is centered around one of the bodies. The event horizon is located on the z-line $z \in [-\sigma, \sigma]$.

As previously mentioned, the expansion is uniform in every compact set of $\{\rho, z\}$ away from the z-line.

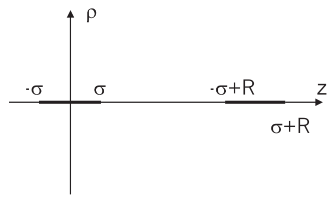

Figure 11.3: $\{\rho, z\}$ plane.

The derivative of v_{Kerr} with respect to ρ^2 has the form

$$\frac{\partial v_{Kerr}}{\partial \rho^2} =$$

$$(4\,\sigma^2\,((-1+2\,p^2)\,z^2+(-1+2\,p^2)\,\rho^2-\sigma^2+$$

$$2\,p^2\,\sigma^2+\sqrt{\rho^2+(z-\sigma)^2}\,\sqrt{\rho^2+(z+\sigma)^2}))/$$

$$\left(\sqrt{\rho^2+(z-\sigma)^2}\,\sqrt{\rho^2+(z+\sigma)^2}\right)/$$

$$((z^2+\rho^2+3\,\sigma^2-\sqrt{\rho^2+(z-\sigma)^2}\,\sqrt{\rho^2+(z+\sigma)^2}+$$

$$2\,p^2\,\sqrt{\rho^2+(z-\sigma)^2}\,\sqrt{\rho^2+(z+\sigma)^2})^2).$$

For small enough rotation $p^2 > 1/2$, the derivative is separated from zero uniformly for all $z \in [-\sigma, \sigma]$ and bounded ρ.

Let's chose a value $z = z_0$, $z_0 \in (-\sigma, \sigma)$ and consider the function v along the line $z = z_0$. We have

$$|v - v_{Kerr}| < \epsilon, \tag{11.15}$$

$$\frac{\partial v_{Kerr}}{\partial \rho^2} > \alpha \tag{11.16}$$

for some ϵ, α and sufficiently large R, uniformly in a neighborhood of the ergosphere.

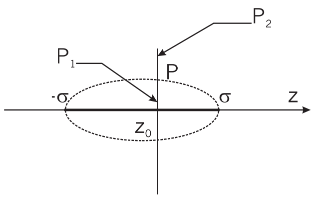

Figure 11.4: Ergosphere.

The function v_{Kerr} vanishes at the point P which is an intersection of the ergosphere with the line $z = z_0$. Because the derivative of v_{Kerr} does not vanish, there are points P_1 and P_2 on the $z = z_0$ line where v_{Kerr} has opposite signs. Then by (11.15), for sufficiently large R, the function v also has opposite signs at the points P_1 and P_2. Therefore, v has to vanish somewhere between P_1 and P_2. Furthermore,by (11.15) and (11.16) the distance between P_1, P_2 and P may be chosen to be the same for different z_0.

The above argument does not work for z_0 in neighborhoods of $z = \pm\sigma$ because the distance from P to the z-axis vanishes as we approach $z = \pm\sigma$. By increasing R we may make those neighborhoods arbitrarily small. Therefore, these are areas where ρ is small.

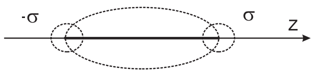

Figure 11.5: Neighborhoods of $z = \pm\sigma$.

So, we use **property 3** of Chapter 9:

$$v = v_0(z) + v_1(z)\rho^2 + o(\rho^2). \qquad (11.17)$$

The expression (11.17) holds uniformly in z and R for every sufficiently large R and in every compact set of z away from the points λ_i. As one can see from the expressions in Chapter 9 the function $v_0(z)$ vanishes at $z = \pm\sigma$ and v_1 does not vanish at $z = \pm\sigma$.

Our next goal is to see that the graph of the solution $\rho^2 = \rho^2(z)$ of the equation

$$v_0(z) + v_1(z)\rho^2 = 0$$

starts at $z = \pm\sigma$ and goes into the area with $z \in [-\sigma, \sigma]$ for sufficiently large R. To see this from $z = \sigma$ it is enough to verify that $-v_0(z)/v_1(z)$ is positive for $z > \sigma$ and negative for $z < \sigma$. As one can see from expression for $v_{z\in I_2}$ and $v_{z\in I_1}$ of **property 3**, $v_0(z)$ is negative in I_2 and positive in I_1. The expression for v_1 is too big and not suitable for print but it is another direct computation to see that for small rotation (precisely $p > 1/2$), $v_1(\sigma)$ is positive. Similarly, $\rho^2(z)$ determined by the above local equation behaves correctly in a neighborhood of every branch point.

We now repeat the argument after (11.16) with the point P on the line $v_0(z) + v_1(z)\rho^2 = 0$ and the z coordinate near one of the branch points (say $\{0, \sigma\}$). We have

$$v = v_0(z) + v_1(z)\rho^2 + o(\rho^2)$$

as $\rho^2 \to 0$, the function $v_0(z)$ vanishes at branch points and the function $v_1(z)$ does not vanish at branch points. Therefore, $v_1(z)$ is separated from zero in a sufficiently small neighborhood of σ. Let $\rho_0(z)$ be a solution of the equation

$$v_0(z) + \rho^2 v_1(z) = 0.$$

Then

$$\rho_0^2(z) = \alpha(\sigma - z) + o(\sigma - z)$$

for some α. Let us position P_1 and P_2 at $\{\rho_0^2(z) \pm C(z-\sigma), z\}$ where C is smaller then α to insure that such P_1 and P_2 are never on the z line in the chosen sufficiently small neighborhood of the branch point $\{0, \sigma\}$. We have

$$v(P_{1,2}) = v(\rho_0^2(z) + C(z-\sigma), z) =$$

$$v_0(z) + v_1(z)(\rho_0^2(z) + C(z-\sigma)) + o(\rho_0^2(z) + C(z-\sigma)) =$$

$$= \pm v_1(z)C(z-\sigma) + o(z-\sigma).$$

So, v has different signs at the points $P_{1,2}$. Therefore, v has a zero between the points P_1 and P_2.

The above argument is a final argument in a chain where we repeatedly said "for sufficiently large R ". So, we have to deal with a possibility that the $o(\rho^2)$ term may be significant as R grows. To determine the significance of $o(\rho^2)$ we compare dominating terms in the expressions for the numerator of v in **properties 2 and 3**. We see that, indeed, the R expansion of the numerator has terms of order R^6 and R^5, but the ρ expansion has the dominating term of order R^4. However, one can see that the R^5 and R^6 vanish at the Kerr's ergosphere and the signature of these terms and the denominator of v at different sides of the ergosphere is the same as the signature of $v_0(z) + \rho^2 v_1(z)$. Therefore, the presence of $o(\rho^2)$ does not change the results of the above argument.

We conclude that there is a solution of $v(\rho^2(z), z) = 0$ for every $z \in [-\sigma, \sigma]$.

Because of the closeness of the derivatives of v and v_{Kerr} for sufficiently large R there are no double zeros.

We next show that even though v does vanish at those ergospheres the functions X and Y of (11.1) and (11.2) are smooth.

Let us consider as an example the Kerr solution. The expressions for the Kerr solution are given in Chapter 5. The function v vanishes at the ergosphere given by the equation

$$pX^2 + qY^2 - 1 = 0, \tag{11.18}$$

where X and Y are prolate spheroidal coordinates introduced in Chapter 5. Let us take, for ease of computation, $p = q = \frac{1}{\sqrt{2}}$. Then the ergosphere becomes the circle in the $\{\rho, z\}$ plane with radius σ, centered at the origin. Make the change of variables $\{\rho, z\} \to \{r, \theta\}$:

$$z = rCos\theta,$$

$$\rho = rSin\theta.$$

In $\{r, \theta\}$ variables the equations (1.6), (1.7) have the form

$$\frac{\partial A}{\partial r} = -\frac{Sin\theta}{v^2}\frac{\partial \Phi}{\partial \theta}, \tag{11.19}$$

$$\frac{\partial A}{\partial \theta} = \frac{Sin\theta}{v^2}\frac{\partial \Phi}{\partial r}, \tag{11.20}$$

and the functions v and Φ have expansions

$$v = \frac{r - \sigma}{2\,\sigma + \sqrt{2}\,\sigma\,\cos(\frac{\theta}{2}) + \sqrt{2}\,\sigma\,\sin(\frac{\theta}{2})} + O((r-\sigma)^3), \tag{11.21}$$

$$\Phi = \frac{\sqrt{2}\,\left(\cos(\frac{\theta}{2}) - \sin(\frac{\theta}{2})\right)}{2 + \sqrt{2}\,\cos(\frac{\theta}{2}) + \sqrt{2}\,\sin(\frac{\theta}{2})} + O((r-\sigma)^2). \tag{11.22}$$

One can verify by direct computation that the above expansions are differentiable.

It follows from (11.19), (11.20), (11.21) and (11.22) that A should have an expansion of the form

$$A = \frac{A_{-1}}{r - \sigma} + O(1).$$

Indeed, v has a zero of first order at $r = \sigma$. So, the RHS of equations (11.19)-(11.20) have poles of second order. Therefore, since these equations are compatible the function A must have a pole of first order. It also follows from (11.19)-(11.20) that A has an expansion in terms of powers of r.

Then it is possible that the singularity $\frac{1}{v}$ in the expression (11.6) may be canceled by the singularity in the second term. A direct computation using the expression for A in Chapter 5 shows that this indeed happens and hence the function X for the Kerr solution is a smooth function.

We now prove that the functions X and Y of (11.1) and (11.2) are indeed smooth around the ergospheres for the two rotating black hole solution. We choose a point at one of the ergospheres. The following argument is a local argument in a neighborhood of the point. We do a change of variables $\{\rho, z\} \to \{v, w\}$ where the v is the real part of the Ernst potential, and w is chosen so that the gradients of v and w with respect to the $\{\rho, z\}$ variables are orthogonal. Since the gradient of v_{Kerr} in $\{\rho, z\}$ variables never vanishes, the gradient of v never vanishes. Furthemore, the gradient is never tangent to the ergosphere. By the general theory of PDEs of first order this property of the gradient and smoothness of v is sufficent to conclude that such a change of variables exists, is smooth and locally non-degenerate.

By **property** 1 of Chapter 9 the function Φ is smooth in the region of interest. The denominator of Φ does not vanish because otherwise the Ernst potential would have a pole which is not possible by **property 4** of Chapter 9. Since Φ is a smooth function is has a local differentiable expansion:

$$\Phi = \Phi_0 + \Phi_1 v + O(v^2), \tag{11.23}$$

where Φ_0 and Φ_1 are functions of w.

As we mentioned in Chapter 1 the equations (1.6) and (1.7) are compatible provided that the Ernst equation is satisfied. Hence, the equations (1.6) and (1.7) have a solution A which is smooth away from $v = 0$. The equations (11.7)-(11.8) are compatible provided that v,Φ solve the Ernst equations and A solves (1.6) and (1.7). Thus, away from $v = 0$, we define smooth functions X, Y according to $(11.6) - (11.8)$. Then the couple $\{X, Y\}$ satisfies (11.1) and (11.2).

The equations (1.6), (1.7), (11.6)-(11.8), (11.1)-(11.2) in $\{v, w\}$ variables have the following form:

$$\frac{\partial A}{\partial v} = \frac{\rho}{v^2\Delta}\left(\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial \rho}\right)^2\right)\frac{\partial \Phi}{\partial w} \tag{11.24}$$

$$\frac{\partial A}{\partial w} = -\frac{\rho}{v^2\Delta}\left(\left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial \rho}\right)^2\right)\frac{\partial \Phi}{\partial v}, \tag{11.25}$$

$$X = \frac{\rho^2}{v} - A^2 v, \tag{11.26}$$

$$v^2\Delta\frac{\partial Y}{\partial v} = 2Av^2\frac{\partial w}{\partial \rho} + \left(A^2v^2 + \rho^2\right)\Delta\frac{\partial \Phi}{\partial v}, \tag{11.27}$$

$$v^2\Delta\frac{\partial Y}{\partial w} = -2Av^2\frac{\partial v}{\partial \rho} + \left(A^2v^2 + \rho^2\right)\Delta\frac{\partial \Phi}{\partial w} + 2Av\rho\left(\left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial \rho}\right)^2\right), \tag{11.28}$$

$$X\frac{\partial^2 X}{\partial v^2}\rho\left(\left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial \rho}\right)^2\right) + X\frac{\partial^2 X}{\partial w^2}\rho\left(\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial \rho}\right)^2\right) +$$

$$X\frac{\partial X}{\partial v}\left(\frac{\partial v}{\partial \rho}+\rho\left(\frac{\partial v}{\partial z}\right)^2+\rho\left(\frac{\partial v}{\partial \rho}\right)^2\right)+X\frac{\partial X}{\partial w}\left(\frac{\partial w}{\partial \rho}+\rho\left(\frac{\partial w}{\partial z}\right)^2+\rho\left(\frac{\partial w}{\partial \rho}\right)^2\right)=$$

$$\left(\frac{\partial X}{\partial v}\right)^2\rho\left(\left(\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial v}{\partial \rho}\right)^2\right)+\left(\frac{\partial X}{\partial w}\right)^2\rho\left(\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial w}{\partial \rho}\right)^2\right)+$$

$$-\left(\frac{\partial Y}{\partial v}\right)^2\rho\left(\left(\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial v}{\partial \rho}\right)^2\right)-\left(\frac{\partial Y}{\partial w}\right)^2\rho\left(\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial w}{\partial \rho}\right)^2\right), \quad (11.29)$$

$$X\frac{\partial^2 Y}{\partial v^2}\rho\left(\left(\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial v}{\partial \rho}\right)^2\right)+X\frac{\partial^2 Y}{\partial w^2}\rho\left(\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial w}{\partial \rho}\right)^2\right)+$$

$$X\frac{\partial Y}{\partial v}\left(\frac{\partial v}{\partial \rho}+\rho\left(\frac{\partial v}{\partial z}\right)^2+\rho\left(\frac{\partial v}{\partial \rho}\right)^2\right)+X\frac{\partial Y}{\partial w}\left(\frac{\partial w}{\partial \rho}+\rho\left(\frac{\partial w}{\partial z}\right)^2+\rho\left(\frac{\partial w}{\partial \rho}\right)^2\right)=$$

$$2\rho\frac{\partial X}{\partial v}\frac{\partial Y}{\partial v}\left(\left(\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial v}{\partial \rho}\right)^2\right)+2\rho\frac{\partial X}{\partial w}\frac{\partial Y}{\partial w}\left(\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial w}{\partial \rho}\right)^2\right). \quad (11.30)$$

where $\Delta = \frac{D(v,w)}{D(\rho,z)}$.

The compatibility condition for (11.24)-(11.25) is of the form

$$\frac{\partial}{\partial w}\left(\frac{f}{v^2}\frac{\partial \Phi}{\partial w}\right)=\frac{\partial}{\partial v}\left(\frac{h}{v^2}\frac{\partial \Phi}{\partial v}\right),$$

where we introduced obvious notations f and h for the coefficients in the equations (11.24) and (11.25). We substitute the expansions $\Phi = \Phi_0 + \Phi_1 v + O(v^2)$, $f = f_0 + O(v)$ and $h = h_0 + O(v)$ into the above equation and obtain

$$O\left(\frac{1}{v^2}\right)=\frac{(-2)}{v^3}(h_0+O(v))(\Phi_1+O(v))+O\left(\frac{1}{v^2}\right).$$

Gathering terms of order v^{-3}, we conclude that we must have $\Phi_1 = 0$ in order for (11.24)-(11.25) to be compatible. Thus, $\Phi = \Phi_0 + O(v^2)$ and according to

(11.24)-(11.25) we have

$$\frac{\partial A}{\partial v} = O\left(\frac{1}{v^2}\right),$$

$$\frac{\partial A}{\partial w} = O\left(\frac{1}{v}\right).$$

Thus, A has to have an expansion

$$A = \frac{A_{-1}}{v} + A_0 logv + O(1), \tag{11.31}$$

where A_{-1} must be non zero and A_0 may or may not be zero.

Using the properties $\frac{\partial \Phi}{\partial v} = O(v)$ and $A = O(1/v)$ with equation (11.27) we see that

$$\frac{\partial Y}{\partial v} = O(1/v).$$

Therefore, the singularity of Y at $v = 0$ is at most

$$Y = O(logv).$$

We desire to show that the function X given by (11.6) is not singular. Let us assume that X is singular and obtain a contradiction. According to (11.26) if X is singular then it has an expansion

$$X = \frac{X_{-1}}{v} + X_0 logv + O(1).$$

We substitute this into (11.29) and observe that the highest order terms are $X\partial_v^2 X$ and $(\partial_v X)^2$. Both of these terms are multiplied by the same nonvanishing coefficient but $X\partial_v^2 X = 2\frac{X_{-1}^2}{v^{-4}} + o(v^{-4})$ and $(\partial_v X)^2 = \frac{X_{-1}^2}{v^{-4}} + o(v^{-4})$. But then the equation (11.29) cannot be satisfied. Therefore, $X_{-1} = 0$. Similarly, considering

(11.29) and $X = O(logv)$ we conclude that X is not singular. But then Y is not singular either according to (11.30). Furthemore, we return to the relationship (11.26) and see that if X is bounded at $v = 0$ then it must be that $A_0 = 0$. Thus, A does not have a logarithmic term and the construction of the couple $\{X, Y\}$ is expandible inside of the ergosphere. This concludes the proof that the pair $\{X, Y\}$ is smooth.

To see that conditions 1 and 2 are satisfied, we note that we are considering smooth functions in a bounded region. The only problem is that X_1 and X_2 in the expression for the $dist(.)$ vanishes on the z-axis. However, the constructed solution $\{X, Y\}$ also vanishes at the z-axis. Again, since we are considering smooth functions we do Taylor expansions and conclude that the zero in the denominator of the expression for $dist(.)$ is compensated by the zero in the numerator.

To see that condition 3 is satisfied we observe that F_3 is the Schwartzchild solution $F_3 = \{X_0, 0\}$. At infinity it has the form $F_3 \to \{\rho, 0\}$. Our solution has the same asymptotics by **the properties 5,6** and **7** of Chapter 9.

This completes the argument that our solution is the unique physical solution for the problem at hand.

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